Topological magnetism

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Part 2: Dynamics

- **Landau-Lifshitz-Gilbert and Thiele equations**
- **Current-driven motion**
- **Dynamics of domain walls and skyrmions**
- **Large-amplitude collective motion excited by electromagnetic fields**

Landau-Lifshitz-Gilbert equation

Classical spin dynamics

Coordinate: $q = \varphi$

Momentum: $p = L_z = \hbar S \cos\theta$ $\qquad L_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}$

Equations of motion:

$$\dot{q} = +\frac{\partial H}{\partial p} \qquad \hbar S \sin\theta\,\dot{\theta} = \frac{\partial H}{\partial \varphi}$$

$$\dot{p} = -\frac{\partial H}{\partial q} \qquad \hbar S \sin\theta\,\dot{\varphi} = -\frac{\partial H}{\partial \theta}$$

Spin precession in a magnetic field:

$$H = -\mu_z H_z = 2\mu_B H_z S \cos\theta$$

$$\boldsymbol{\mu} = -2\mu_B \mathbf{S}$$

$$\hbar\dot{\varphi} = 2\mu_B H_z = \hbar\frac{eH_z}{m_e c}$$

Classical spin dynamics

Lagrangian: $$L = p\dot{q} - H = \hbar S \cos\theta\, \dot{\varphi} - H(\theta, \varphi)$$

Action: $$A_S = \int dt\, L$$

Euler-Lagrange equation: $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

$$q = \varphi: \qquad \hbar S \sin\theta\, \dot{\theta} = \frac{\partial H}{\partial \varphi}$$

$$q = \theta: \qquad \hbar S \sin\theta\, \dot{\varphi} = -\frac{\partial H}{\partial \theta}$$

Problem 1: $$\hbar \dot{\mathbf{S}} = \mathbf{S} \times \left(-\frac{\partial H}{\partial \mathbf{S}}\right)$$

Dissipation

Friction force: $$\dot{p} = -\frac{\partial H}{\partial q} + f \qquad f = -\alpha \dot{q}$$

Many degrees of freedom: $$f_i = -\alpha_{ik} \dot{q}_k$$

Onsager relations: $$\alpha_{ik} = \alpha_{ki} \qquad f_i = -\frac{\partial F}{\partial \dot{q}_i}$$

Dissipation function: $$F = \frac{1}{2} \dot{q}_i \alpha_{ik} \dot{q}_k$$

Equation of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} - \frac{\partial F}{\partial \dot{q}_i}$$

Energy decrease:

$$\frac{dH}{dt} = -2F$$

$\boxed{\text{Problem 2 Proof it}}$

Landau-Lifshitz-Gilbert equation

Dissipation function for spin:

$$F = \frac{\alpha\hbar}{2S}(\dot{\mathbf{S}} \cdot \dot{\mathbf{S}}) = \frac{\alpha\hbar}{2S}\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2\right)$$

Problem 3: $$\hbar\dot{\mathbf{S}} = \mathbf{S} \times \left(-\frac{\partial H}{\partial \mathbf{S}}\right) - \frac{\alpha\hbar}{S}\,\mathbf{S} \times \dot{\mathbf{S}}$$

Magnetic moment: $$\boldsymbol{\mu} = \mu\mathbf{m} = 2\mu_B S\mathbf{m} \qquad \mathbf{m} = -\mathbf{n}$$

LLG equation: $$\dot{\mathbf{m}} = -\gamma[\mathbf{m} \times \mathbf{H}_{\mathrm{eff}}] - \alpha\,[\mathbf{m} \times \dot{\mathbf{m}}]$$

Effective field:

$$\mathbf{H}_{\mathrm{eff}} = -\frac{\partial H}{\partial \boldsymbol{\mu}}$$

Gyromagnetic ratio:

$$\gamma = \frac{2\mu_B}{\hbar} = \frac{e}{m_e c}$$

Continuum model

Problem 4:

Show that at length scales much larger than the lattice constant a the spin model of the cubic Heisenberg ferromagnet

$$H = -J_1 \sum_{n,a} (\mathbf{S}_n \cdot \mathbf{S}_{n+a})$$

is equivalent to $$H = \frac{J}{2} \int dV (\partial_a \mathbf{m} \cdot \partial_a \mathbf{m}) + E_0 \qquad J = \frac{J_1}{a}$$

Effective field: $$\mathbf{H}_{\text{eff}} = -\frac{\delta H}{\delta \mathbf{M}}$$ **M is the magnetization**

$$\delta H = -\int dV \mathbf{H}_{\text{eff}} \delta \boldsymbol{M}$$

Landau-Lifshitz-Gilbert equation

Landau-Lifshitz-Gilbert equation:

$$\dot{\mathbf{m}} = -\gamma\,\mathbf{m}\times\mathbf{H}_{eff} + \alpha\,\mathbf{m}\times\dot{\mathbf{m}}$$

$$\mathbf{m} = \frac{\mathbf{M}}{M_0} \uparrow\downarrow \mathbf{S} \qquad \mathbf{H}_{eff} = -\frac{\delta H}{\delta \mathbf{M}} \qquad \gamma = \frac{g\mu_B}{\hbar}$$

Dimensionless LLG equation: $t \rightarrow t\,\dfrac{M_0}{\gamma}$

$$\dot{\mathbf{m}} = \mathbf{m}\times\frac{\delta H}{\delta \mathbf{m}} + \alpha\,\mathbf{m}\times\dot{\mathbf{m}}$$

Thiele equations

Thiele equations

O. A. Tretiakov et al., PRL ***100***, *127204* (2008)

Collective variables: $\{\xi_i\} \quad i = 1, 2, \ldots, N$

$$\mathbf{m}(\mathbf{x}, t) = \mathbf{m}(\mathbf{x}, \{\xi_i(t)\})$$

$$\boxed{\alpha \Gamma_{ij} \dot{\xi}_j - G_{ij} \dot{\xi}_j = F_i}$$

$$\Gamma_{ij} = +\Gamma_{ji} = \sum_{\mathbf{x}} \frac{\partial \mathbf{m}}{\partial \xi_i} \cdot \frac{\partial \mathbf{m}}{\partial \xi_j} \qquad G_{ij} = -G_{ji} = \sum_{\mathbf{x}} \mathbf{m} \cdot \frac{\partial \mathbf{m}}{\partial \xi_i} \times \frac{\partial \mathbf{m}}{\partial \xi_j}$$

$$F_i = -\sum_{\mathbf{x}} \frac{\partial H}{\partial \mathbf{m}} \cdot \frac{\partial \mathbf{m}}{\partial \xi_i} = -\frac{\partial H}{\partial \xi_i}$$

Thiele equations

Problem 5

Derive Thiele equations, $\alpha \Gamma_{ij} \dot{\xi}_j - G_{ij} \dot{\xi}_j = F_i$, using

the LLG equation: $\dot{\mathbf{m}} = \mathbf{m} \times \dfrac{\delta H}{\delta \mathbf{m}} + \alpha \mathbf{m} \times \dot{\mathbf{m}}$

Hint: Multiply both parts of the LLG equation by $\left[\mathbf{m} \times \dfrac{\partial \mathbf{m}}{\partial \xi_i} \right]$

and sum over **x**.

Domain wall

$$H = \int dx \left[\frac{J}{2} \left(\frac{d\mathbf{m}}{dx} \right)^2 + \frac{K_x}{2} m_x^2 + \frac{K_y}{2} m_y^2 - MHm_z \right]$$

Energy
per unit area

$$\mathbf{m} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

Static domain wall: $H = 0$

Problem 6
Proof it

$$\phi = 0 \quad (K_y > K_x > 0)$$

$$\cos\theta_0 = \tanh\frac{x - X}{w_0} \qquad w_0 = \sqrt{\frac{J}{K_x}}$$

Moving wall (exact solution)

*L.R. Walker et al. JAP **45**, 5406 (1974)*

Domain wall profile:

$$\cos\theta = \tanh\left(\frac{x - Vt}{w}\right)$$

DW width:

$$w = \sqrt{\frac{J}{K_x + (K_y - K_x)\sin^2\phi}}$$

DW rotation:

$$\sin 2\phi = \frac{2MH}{\alpha(K_y - K_x)} = \frac{H}{H_c}$$

DW velocity:

$$V = -\frac{MHw}{\alpha}$$

Moving DW (Thiele equations)

Collective variables: (X, ϕ) $\phi \ll 1$

Thiele:

$$\boxed{\begin{aligned} \alpha\Gamma_{XX}\dot{X} - G_{X\phi}\dot{\phi} &= -2MH \\ G_{X\phi}\dot{X} + \alpha\Gamma_{\phi\phi}\dot{\phi} &= -2(K_y - K_x)w_0\phi \end{aligned}}$$

Solution: $\boxed{\dot{X} = -\frac{MHw_0}{\alpha}}$ $\boxed{\phi = \frac{MH}{(K_y - K_x)\alpha}}$

Problem 7:

$$G_{X\phi} = \int_{-\infty}^{+\infty} dx\, \mathbf{m} \cdot \frac{\partial \mathbf{m}}{\partial X} \times \frac{\partial \mathbf{m}}{\partial \phi} = 2$$

$$\Gamma_{XX} = \int_{-\infty}^{+\infty} dx\, \mathbf{m} \cdot \frac{\partial \mathbf{m}}{\partial X} \times \frac{\partial \mathbf{m}}{\partial X} = \frac{2}{w_0} \qquad \Gamma_{\phi\phi} = \int_{-\infty}^{+\infty} dx\, \mathbf{m} \cdot \frac{\partial \mathbf{m}}{\partial \phi} \times \frac{\partial \mathbf{m}}{\partial \phi} = 2w_0$$

Problem 9

(a) Show that an effective Lagrangian, L, and dissipation function, F, describing the DW motion have the form

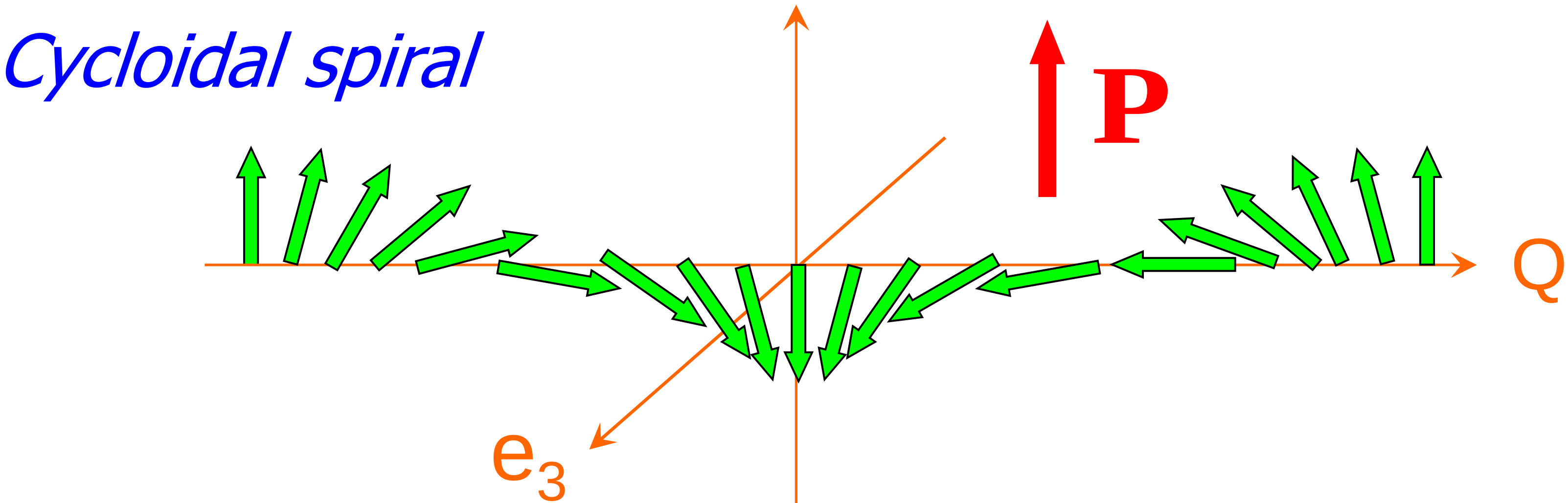
$$L = G_{X\phi} X\dot{\phi} - \left(K_y - K_x\right) w_0 \phi^2 - 2MHX$$

$$F = \frac{\alpha}{2}\left(\Gamma_{XX}\dot{X}^2 + \Gamma_{\phi\phi}\dot{\phi}^2\right)$$

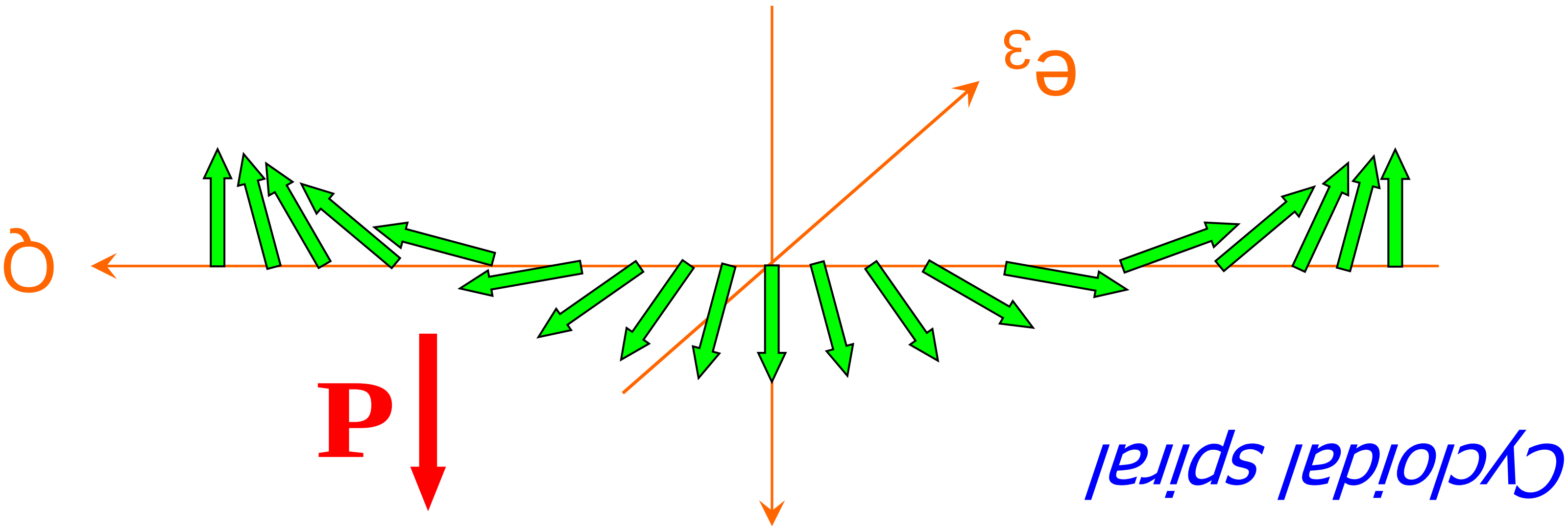
(b) Show that the DW velocity can be found using energy conservation.

Unidirectional motion of domain walls in oscillating electromagnetic fields

Breaking of inversion symmetry by spin ordering

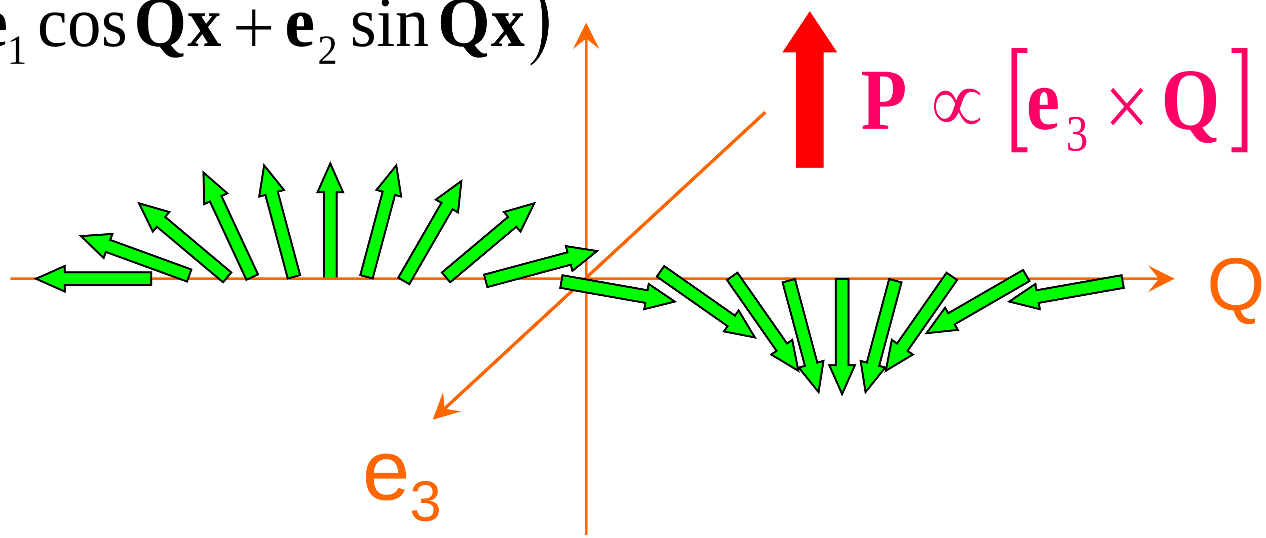


Inversion I: $(x,y,z) \rightarrow (-x,-y,-z)$



Polarization induced by spiral

$$\mathbf{M} = M\left(\mathbf{e}_1 \cos \mathbf{Qx} + \mathbf{e}_2 \sin \mathbf{Qx}\right)$$



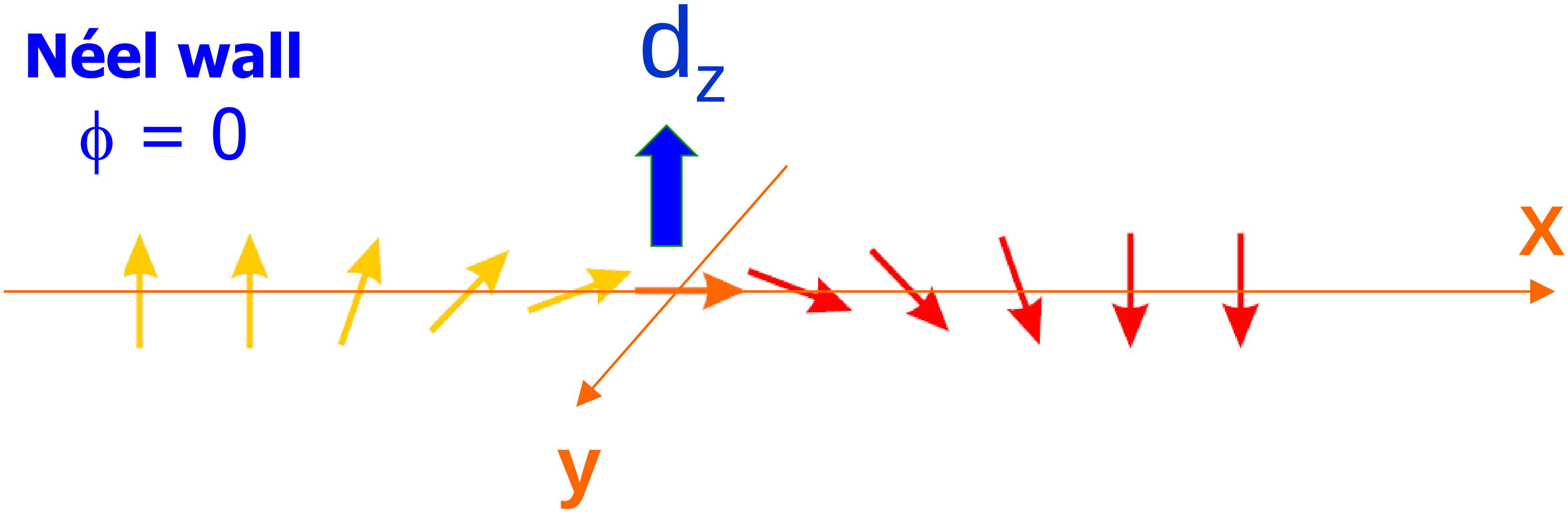
$$F_{me} = -\lambda E_z (M_z \partial_x M_x - M_x \partial_x M_z)$$

Inverse DM mechanism

$$P_z = \frac{\partial F_{me}}{\partial E_z} = \lambda (M_z \partial_x M_x - M_x \partial_x M_z)$$

Bary'akhtar et al, JETP Lett 37, 673 (1983) , Stefanovskii et al, Sov. J. Low Temp. Phys. 12, 478(1986), H. Katsura et al PRL 95, 057205 (2005), Sergienko et al PRB 73, 094434 (2006), M.M. PRL 96, 067601 (2006)

Electric dipole moment of magnetic domain wall



Electric polarization $$P_z = -\lambda \frac{d\theta}{dx} \cos\phi$$

Electric dipole moment is independent of dw shape

$$d_z = \pm\pi\lambda \cos\phi$$

Moving DW (Thiele equation)

Collective variables: $\left(X(t), \phi(t)\right)$

$$\boxed{\begin{aligned} &\alpha \Gamma_{XX} \dot{X} - G_{X\phi} \dot{\phi} = -\frac{\partial E}{\partial X} = 0 \\ &G_{X\phi} \dot{X} + \alpha \Gamma_{\phi\phi} \dot{\phi} = -\frac{\partial E}{\partial \phi} = E_z(t) D \sin\phi \end{aligned}}$$

$$G_{X\phi} = \int_{-\infty}^{+\infty} dx\, \mathbf{m} \cdot \frac{\partial \mathbf{m}}{\partial X} \times \frac{\partial \mathbf{m}}{\partial \phi} = 2$$

$$\Gamma_{XX} = \int_{-\infty}^{+\infty} dx \frac{\partial \mathbf{m}}{\partial X} \cdot \frac{\partial \mathbf{m}}{\partial X} = \frac{2}{w} \qquad \Gamma_{\phi\phi} = \int_{-\infty}^{+\infty} dx \frac{\partial \mathbf{m}}{\partial \phi} \cdot \frac{\partial \mathbf{m}}{\partial \phi} = 2w$$

Heuristic argument

Spins rotate with the frequency ω $\phi = \omega t$

$d_z(t) = D\cos(\omega t)$ **follows** $E_z(t) = E_0\cos(\omega t)$

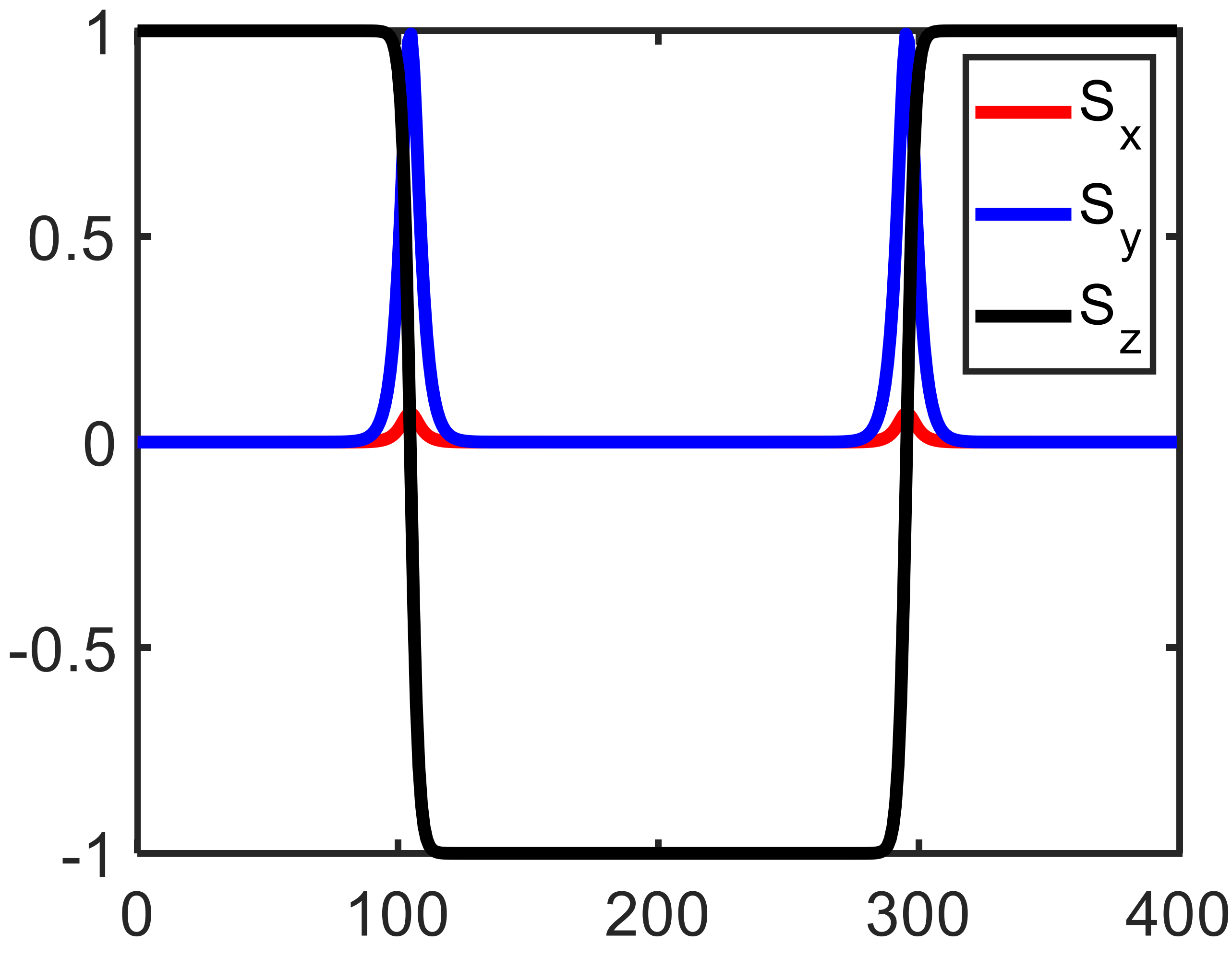
Domain wall moves with the speed

$$\dot{X} = \frac{G_{X\phi}\dot{\phi}}{\alpha\Gamma_{XX}} = \frac{\omega w}{\alpha}$$

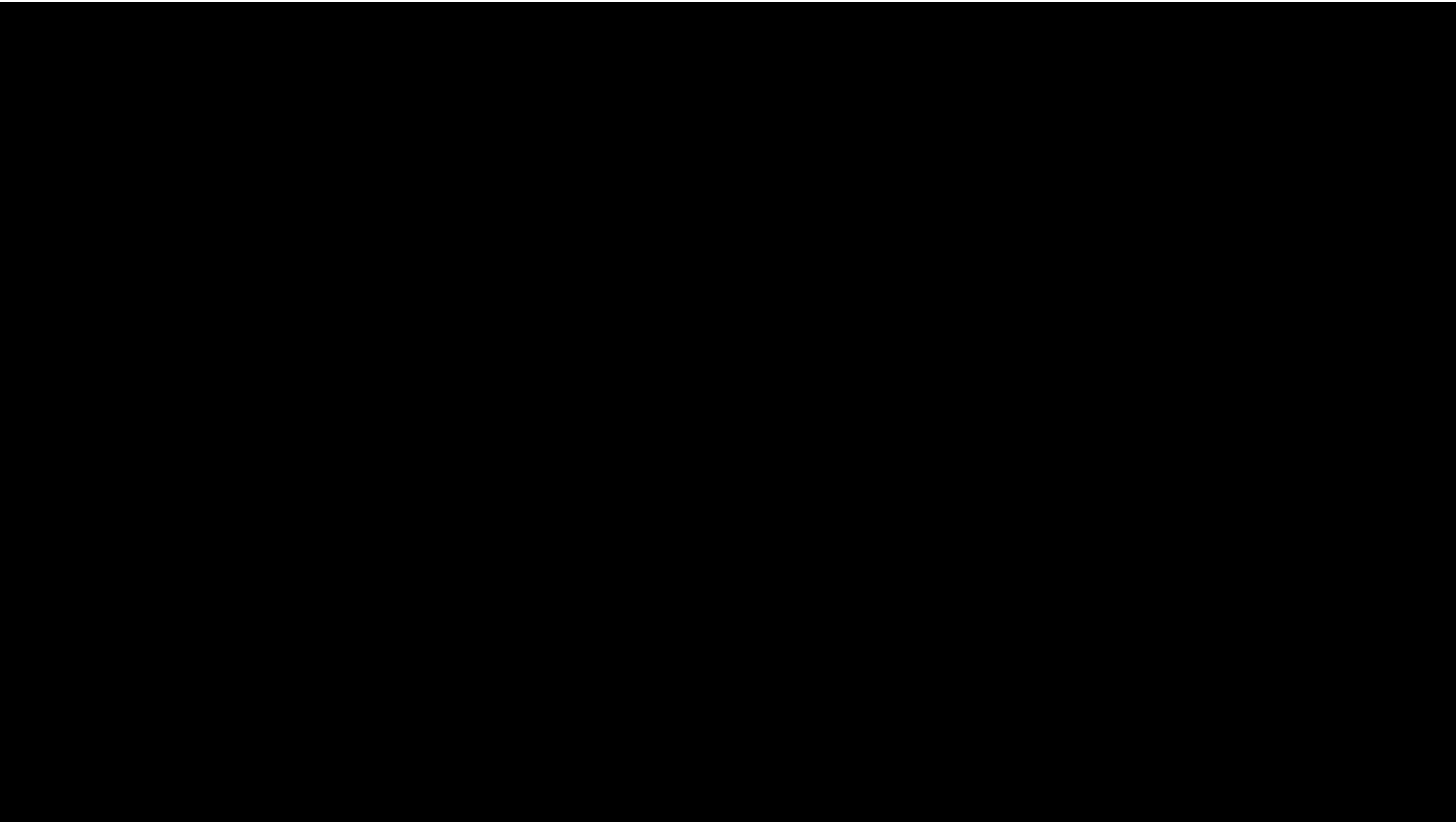
Exact solution gives oscillations

$$\cos\phi = \tanh(A\omega t + B)$$

Domain walls



DW motion in oscillating E||z



Electromagnetic field
+ in-plane anisotropy

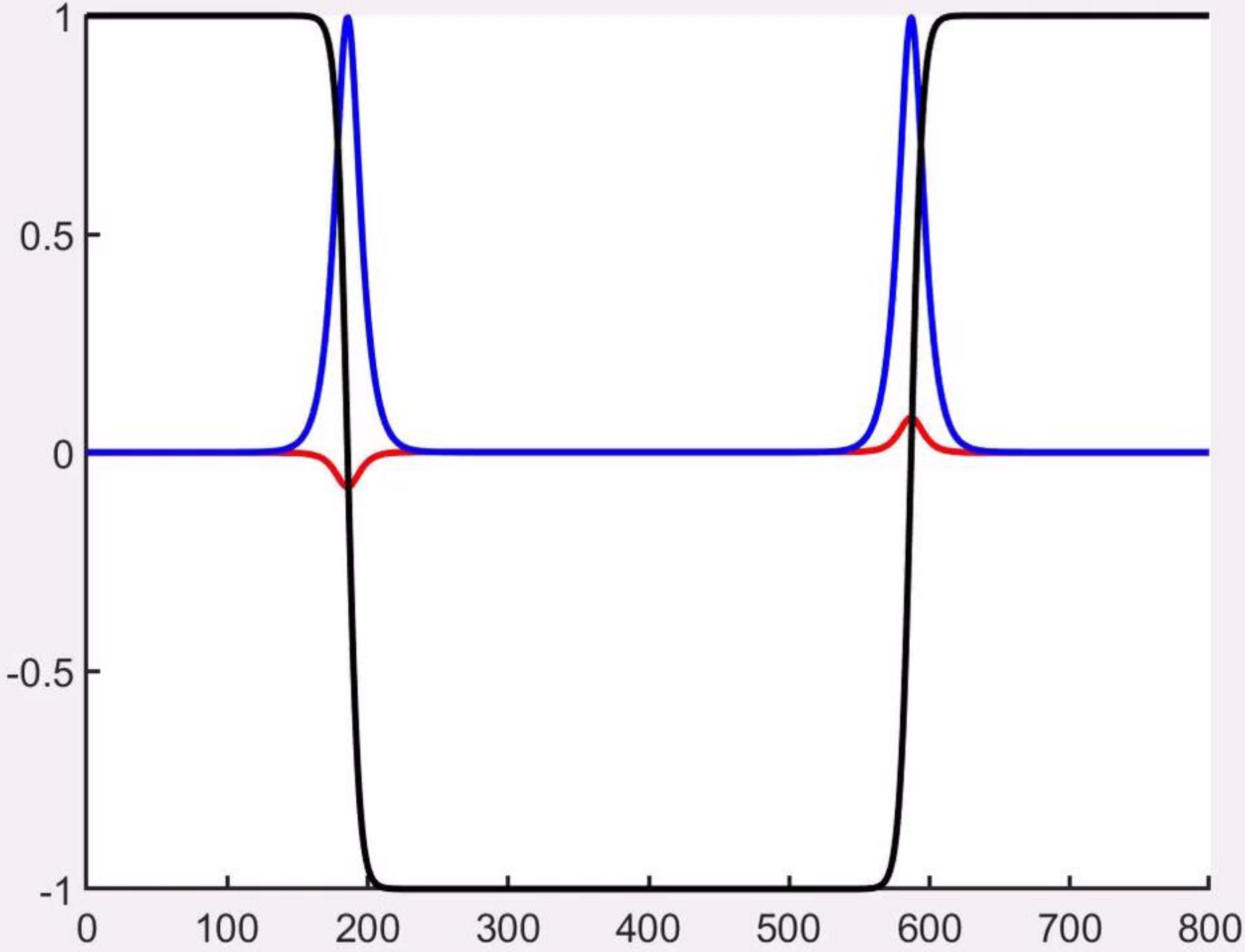
$$E_z(t) = E_0 \cos(\omega t) \qquad H_y(t) = H_0 \sin(\omega t)$$

$$E_{ani} = -\frac{K_z}{2} m_z^2 + \frac{K_y}{2} m_y^2$$

$$\frac{d\phi}{d\tau} = -\alpha_E \cos\tau \cos\phi + \alpha_H \sin\tau \sin\phi + \alpha_K \cos 2\phi$$

$$\tau = \omega t$$

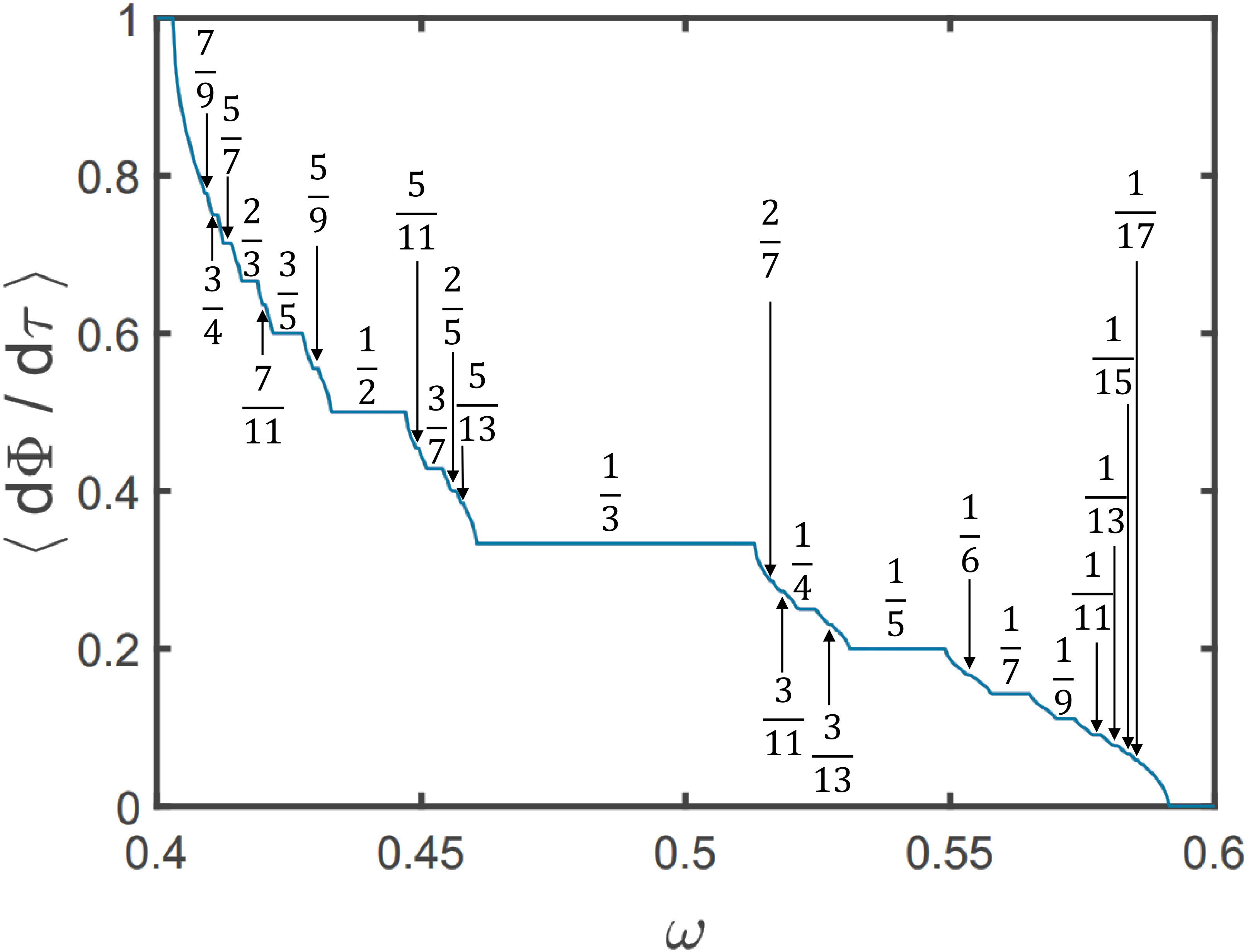
$$\left\langle \frac{d\phi}{d\tau} \right\rangle \approx \mathrm{sign}(\alpha_E \alpha_H) \sqrt{(\alpha_E \alpha_H)^2 - \alpha_K^2}$$



1
0.5
0
-0.5
-1
0
100
200
300
400
500
600
700
800

Devil staircase

M. Parodi, S. Artyukhin and MM, to be published



Skyrmion dynamics

Thiele equations for Skyrmion

Center-of-mass coordinates:

$$\mathbf{m}(\mathbf{x},t) = \mathbf{m}(\mathbf{x} - \mathbf{R}(t)) \qquad \mathbf{R} = (X, Y)$$

$$\boxed{\begin{aligned} \alpha\Gamma\,\dot{\mathrm{X}} - G\,\dot{\mathrm{Y}} &= F_X \\ G\,\dot{\mathrm{X}} + \alpha\Gamma\,\dot{\mathrm{Y}} &= F_Y \end{aligned}}$$

$$G = G_{XY} = -G_{YX} = \sum_{\mathbf{x}} \mathbf{m} \cdot \frac{\partial \mathbf{m}}{\partial x} \times \frac{\partial \mathbf{m}}{\partial y} = 4\pi Q$$

$$\Gamma = \Gamma_{XX} = \sum_{\mathbf{x}} \frac{\partial \mathbf{m}}{\partial x} \cdot \frac{\partial \mathbf{m}}{\partial x} = \Gamma_{YY} = \sum_{\mathbf{x}} \frac{\partial \mathbf{m}}{\partial y} \cdot \frac{\partial \mathbf{m}}{\partial y}$$

Gyrotropic dynamics of skyrmions

$$F_X = 0$$

$$\alpha\Gamma\dot{\mathrm{X}} - G\dot{\mathrm{Y}} = 0$$

$$G\dot{\mathrm{X}} + \alpha\Gamma\dot{\mathrm{Y}} = F_Y$$

$$\dot{Y} = \frac{\alpha\Gamma}{(\alpha\Gamma)^2 + G^2} F_Y \approx \frac{\alpha\Gamma}{G^2} F_Y$$

$$\dot{X} = \frac{G}{(\alpha\Gamma)^2 + G^2} F_Y \approx \frac{F_Y}{G}$$

$$\frac{\dot{\mathrm{Y}}}{\dot{\mathrm{X}}} = \alpha\frac{\Gamma}{G}$$

Skyrmion in inhomogeneous magnetic field

∇H_z

Problem 9

Show that the effective Lagrangian, L, and dissipation function, F, describing skyrmion dynamics have the form

$$L = GX\dot{Y} - H(X, Y)$$

$$F = \frac{\alpha\Gamma}{2}\left(\dot{X}^2 + \dot{Y}^2\right)$$

Force: $$F_X = -\frac{\partial H}{\partial X} \qquad F_Y = -\frac{\partial H}{\partial Y}$$

Skyrmion Hall effect

*A. P. Malozemoff, JAP **44**, 5080 (1973)*

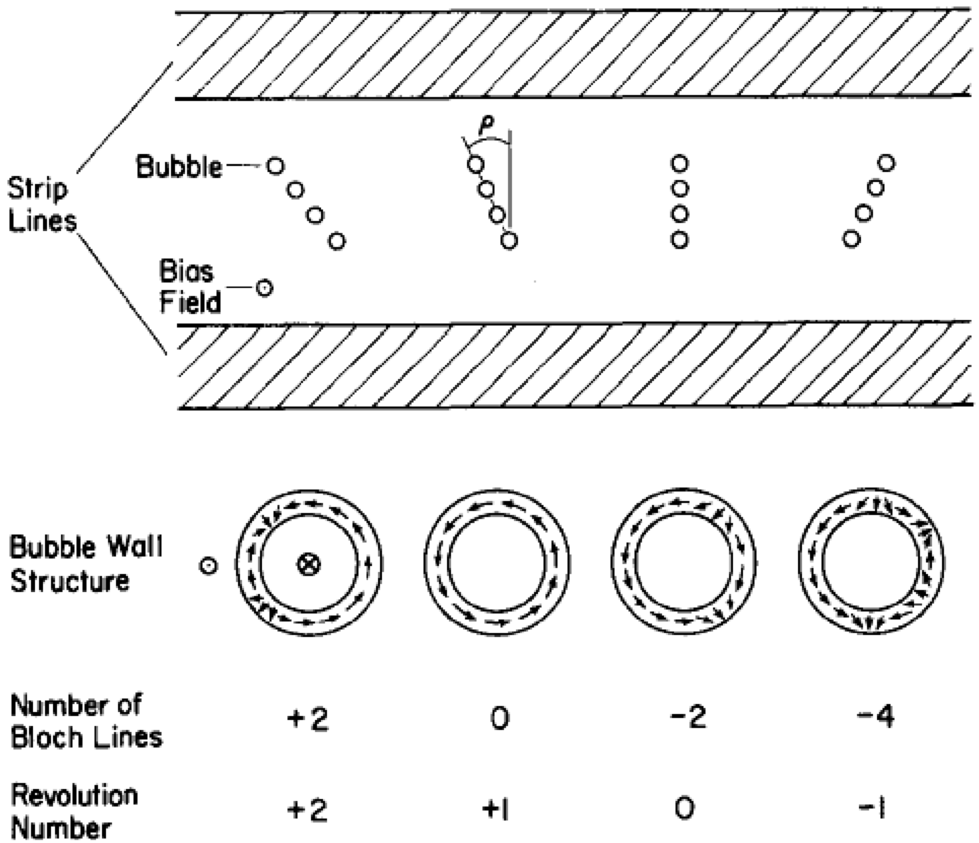
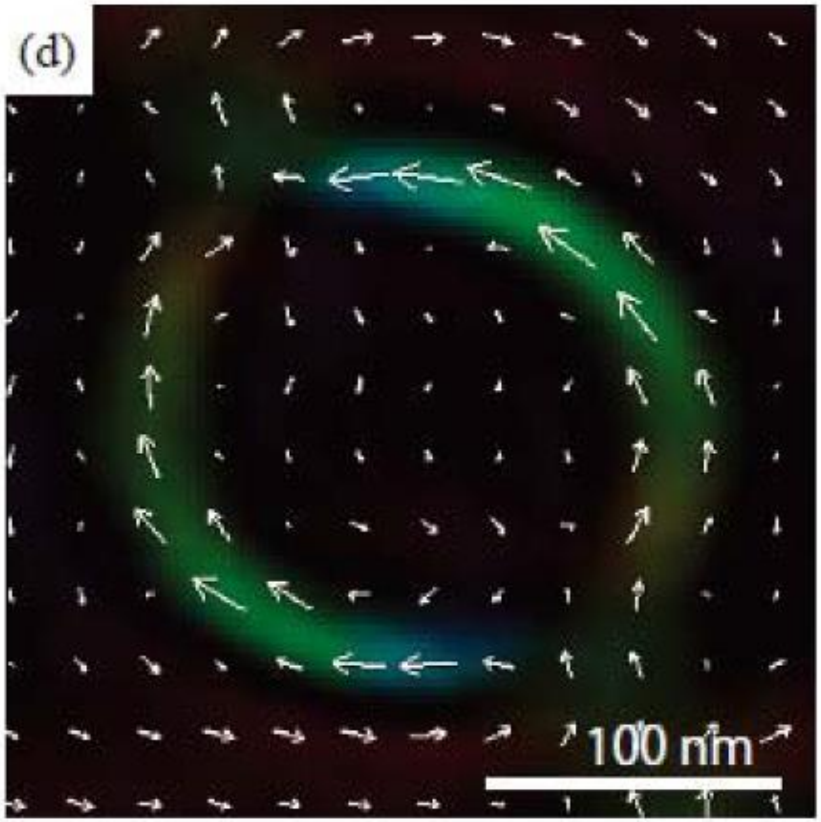


FIG. 1. Schematic illustration of bubble propagation experiment. Equal currents are pulsed in the same direction down two strip lines, creating a gradient field, and bubble motion is observed in multiple photographic exposures. The presumed wall structure, number of BL's, and revolution number are indicated below each of the four bubbles deflecting in different directions.

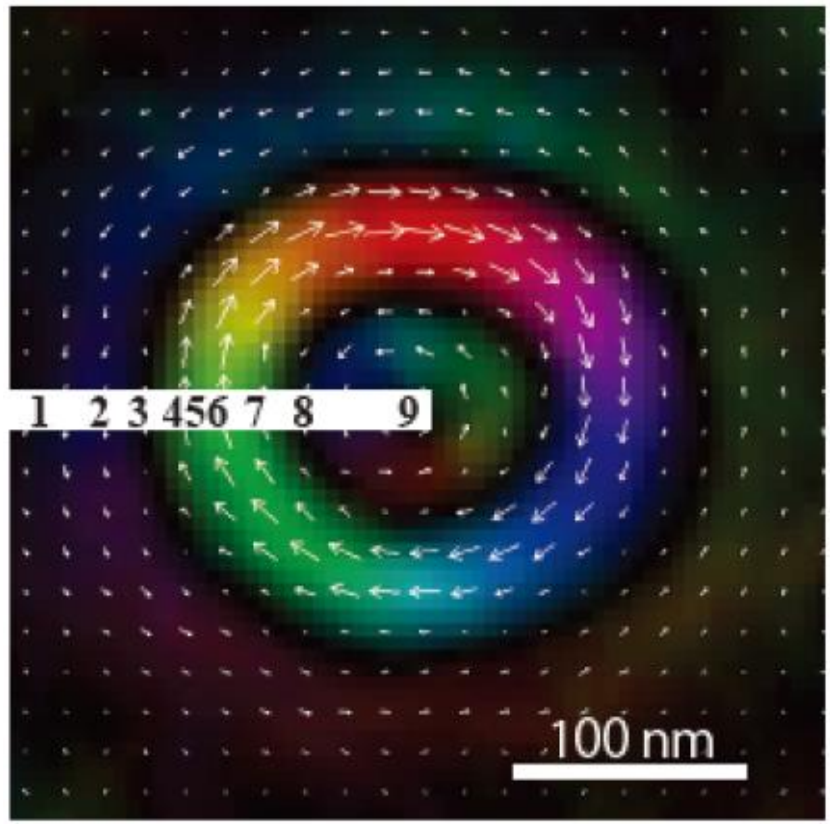
Dynamics of bubbles in non-uniform magnetic field

Q = 0



moves straight

Q = 1



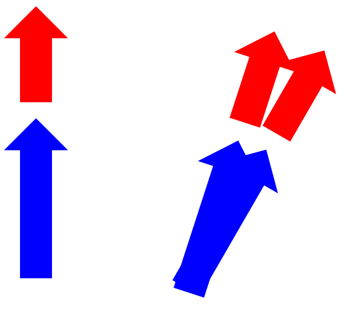
moves askew

Current-driven dynamics of spin textures

LLG equation in presence of electric current

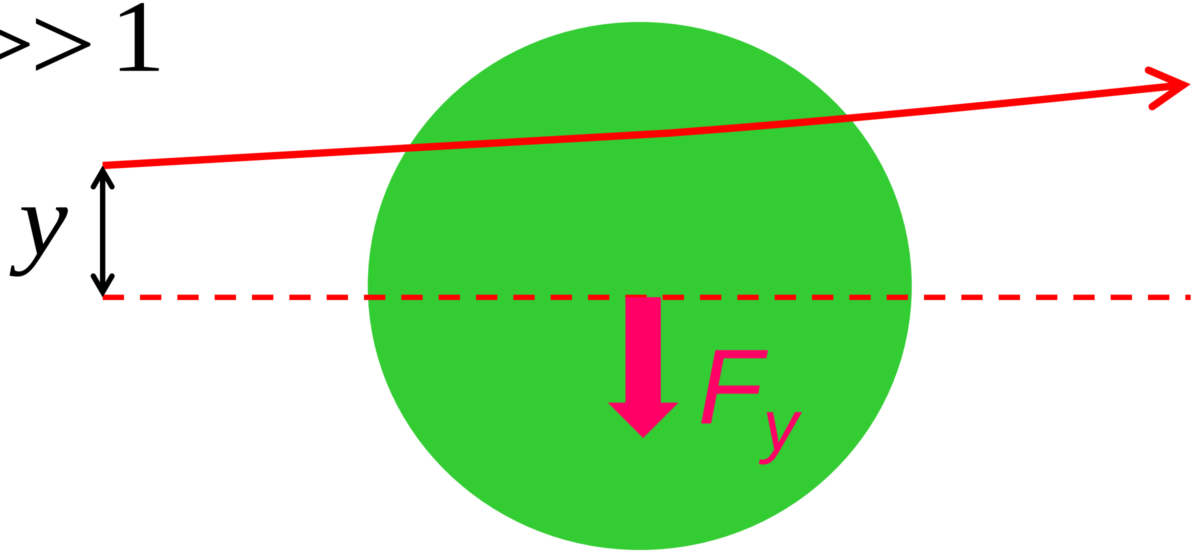
$$\dot{\mathbf{m}} = \mathbf{m}\times\frac{\partial H}{\partial \mathbf{m}} + \alpha\,\mathbf{m}\times\dot{\mathbf{m}} + (\mathbf{j}\cdot\nabla)\mathbf{m} - \beta\,\mathbf{m}\times(\mathbf{j}\cdot\nabla)\mathbf{m}$$

spin-transfer torque



Magnus force and current-driven motion of skyrmions

$kR_s >> 1$



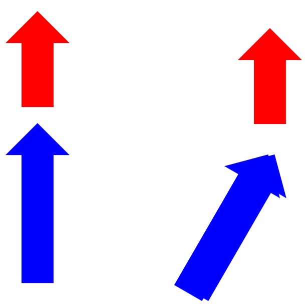
$$\Delta p_y = -\frac{e}{c}\int dx h_z(x,y)$$

$$F_y = -n_e \mathrm{v}_x \int dy\, \Delta p_y \propto -Q j_x \qquad \dot{X} \approx \frac{F_y}{4\pi Q} \propto -j_x$$

LLG equation in presence of electric currents

$$\dot{\mathbf{m}} = \mathbf{m} \times \frac{\partial H}{\partial \mathbf{m}} + \alpha\, \mathbf{m} \times \dot{\mathbf{m}} + (\mathbf{j} \cdot \nabla)\mathbf{m} - \beta\, \mathbf{m} \times (\mathbf{j} \cdot \nabla)\mathbf{m}$$

β-term

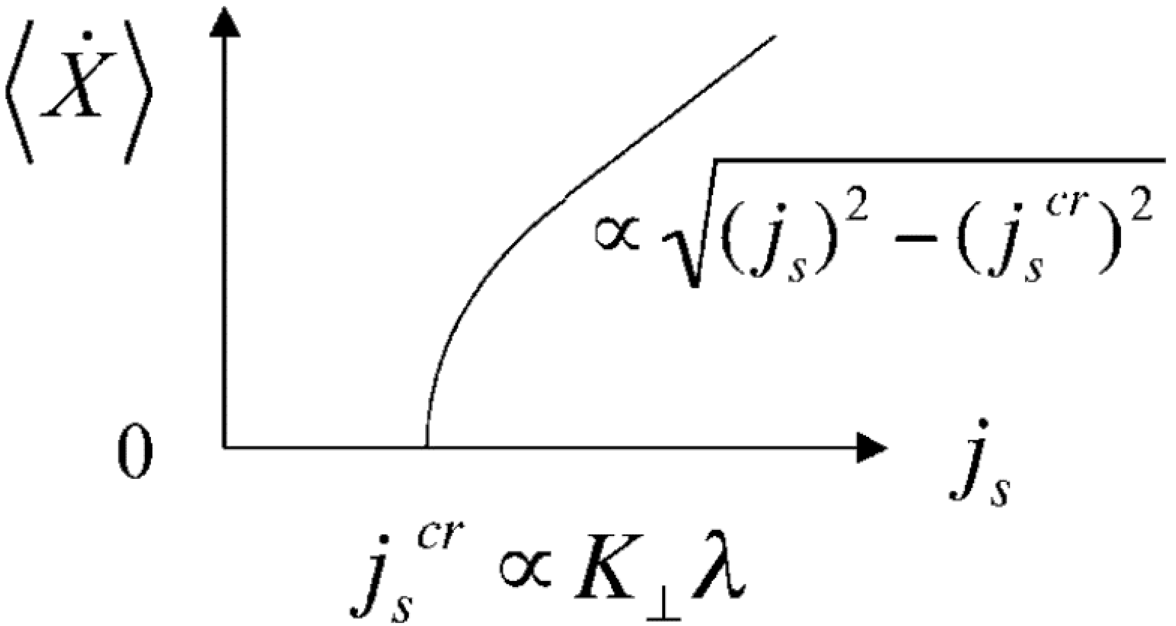


Problem 10

Show that Thiele equations for DW in presence of electric current have the form

$$\begin{cases} \alpha\Gamma_{\phi\phi}\dot{\phi} + G_{X\phi}\dot{X} = -\dfrac{\partial H}{\partial \phi} - G_{X\phi} j_x \\ -G_{X\phi}\dot{\phi} + \alpha\Gamma_{XX}\dot{X} = -\beta\Gamma_{XX} j_x \end{cases}$$

Intrinsic pinning



For $\beta = 0$ the DW cannot move without changing its shape

*G. Tatara & H. Kohno, PRL **92**, 086601 (2004)*

Current-driven Skyrmion dynamics

Thiele equations :

$$\begin{cases} \alpha\Gamma\dot{X} - G\dot{Y} = -\beta\Gamma j_x \\ G\dot{X} + \alpha\Gamma\dot{Y} = -G j_x \end{cases}$$

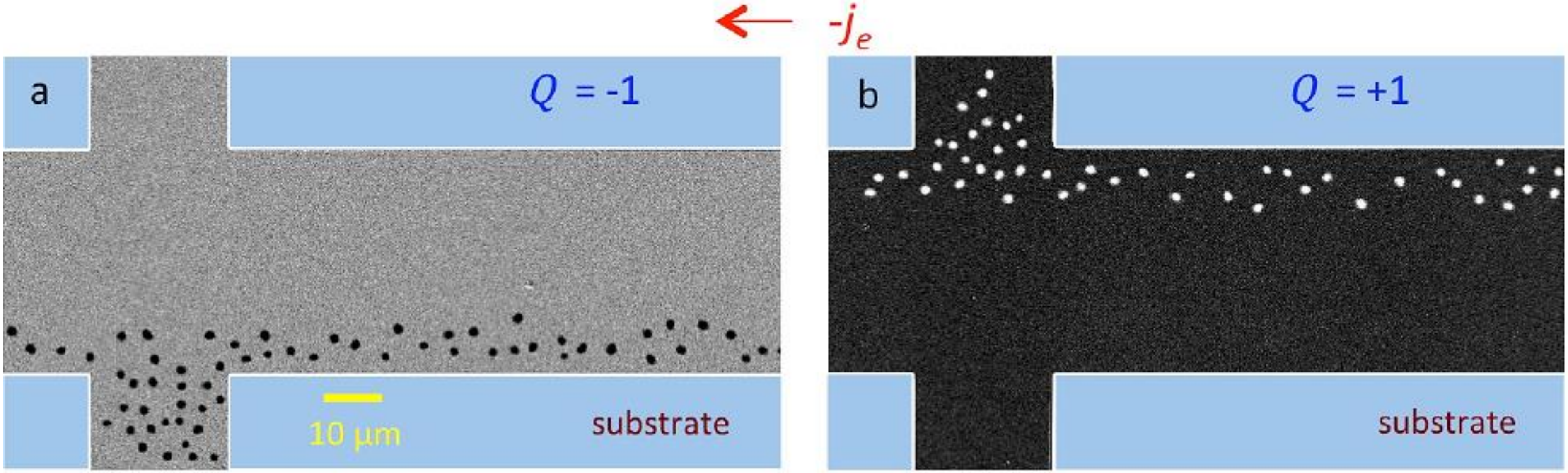
$$\dot{X} = -\frac{G^2 + \alpha\beta\Gamma^2}{G^2 + (\alpha\Gamma)^2} j_x \approx -j_x \qquad \dot{Y} = \frac{(\beta - \alpha)\Gamma G}{G^2 + (\alpha\Gamma)^2} j_x$$

Skyrmion Hall Effect :

$$\frac{V_y}{V_x} \approx (\alpha - \beta)\frac{\Gamma}{G}$$

Skyrmion Hall effect

*W. Jiang at al. Nature Physics **13**, 162 (2017)*



Summary

- **Thiele equations for collective degrees of freedom help to understand dynamics of topological defects**
- **Topology affects dynamics**
- **Domain walls and skyrmions can be set into motion by electrical currents and electromagnetic fields**