Topological magnetism

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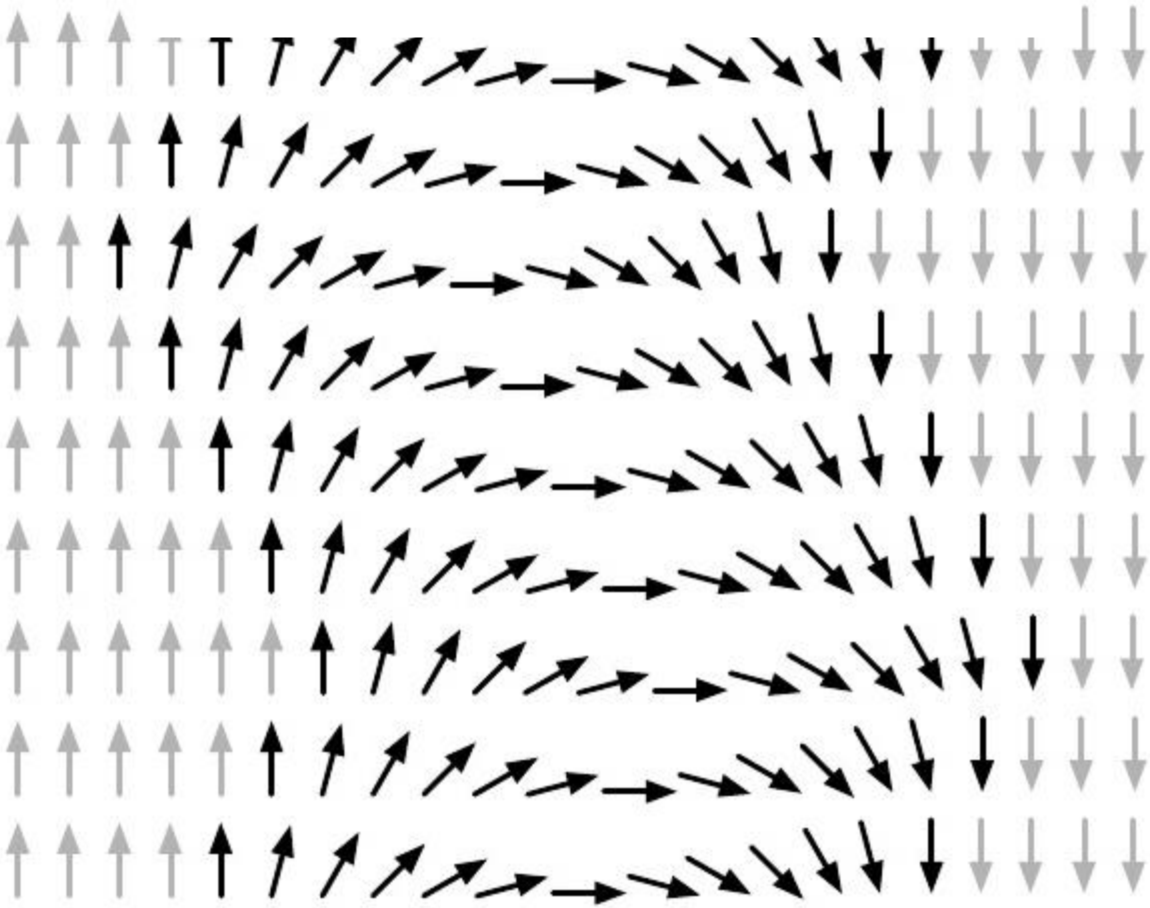
Part 1: Topology

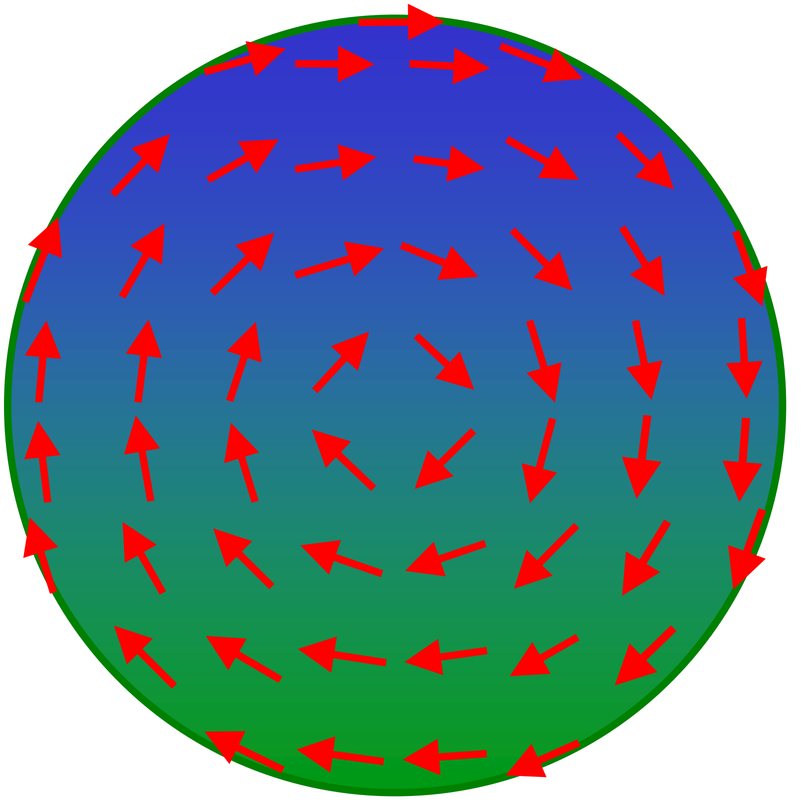
- **Topological defects in magnets**
- **Topological numbers**
- **Effective electromagnetic fields**
- **Stability**

Topological defects

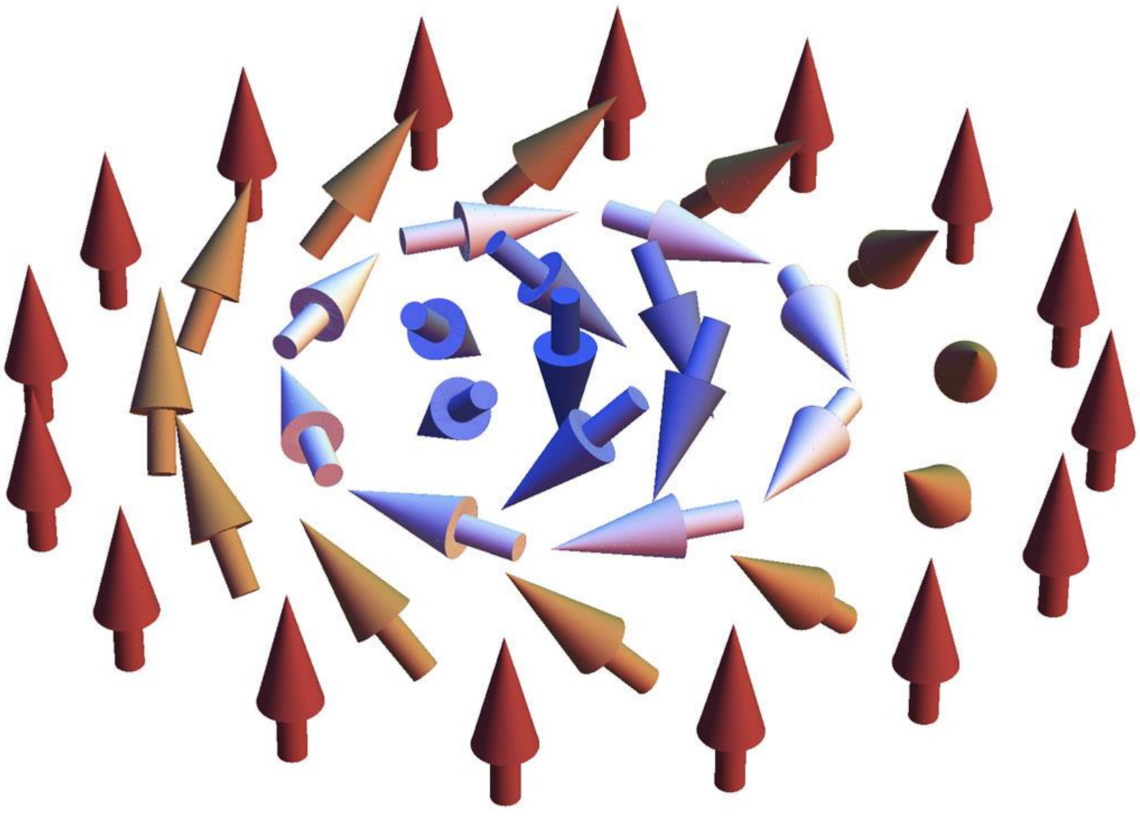
Domain wall



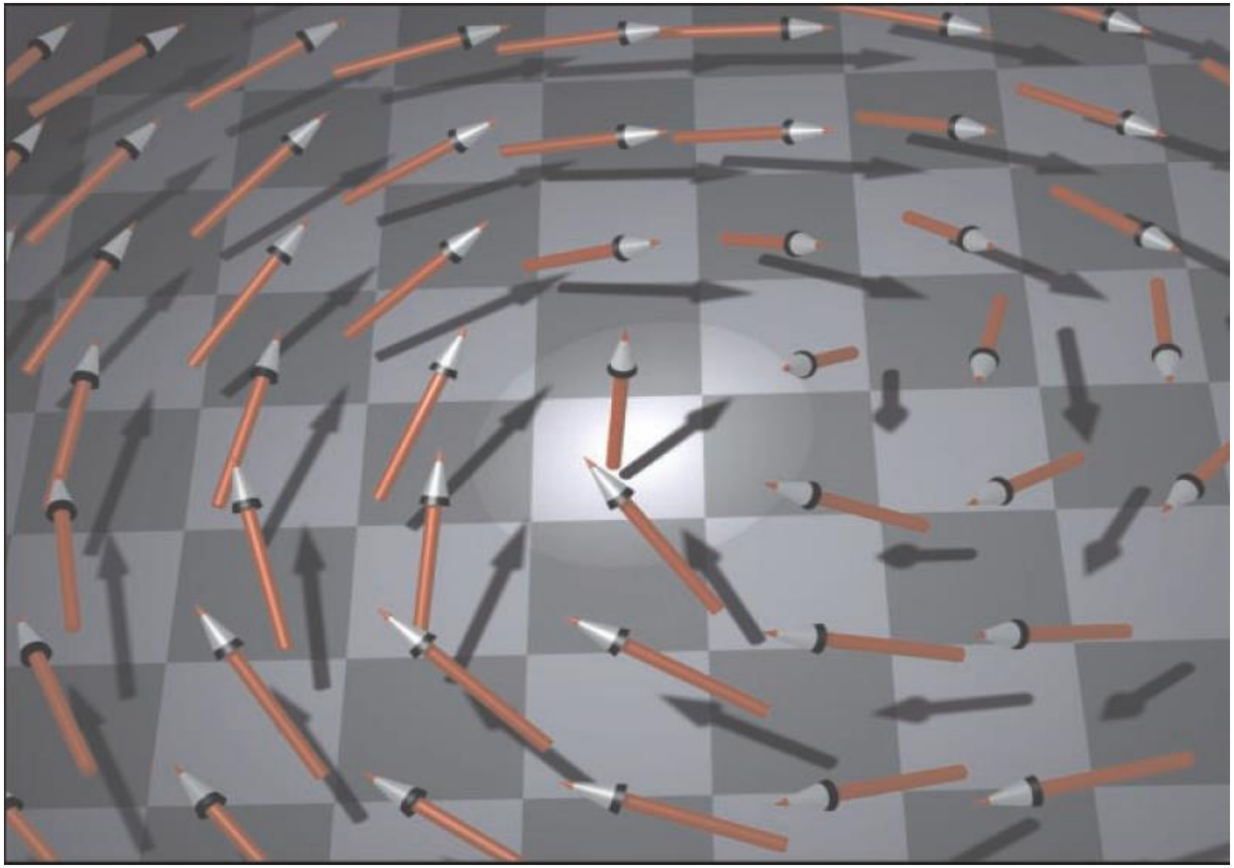
Vortex



Magnetic skyrmion

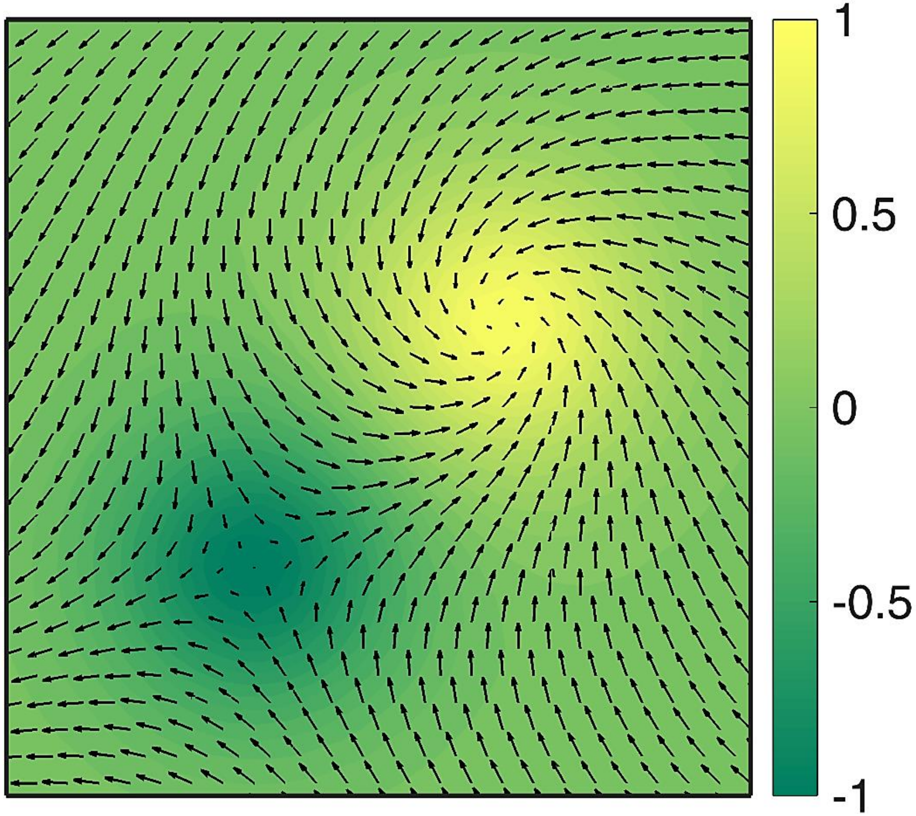


Magnetic vortex (meron)

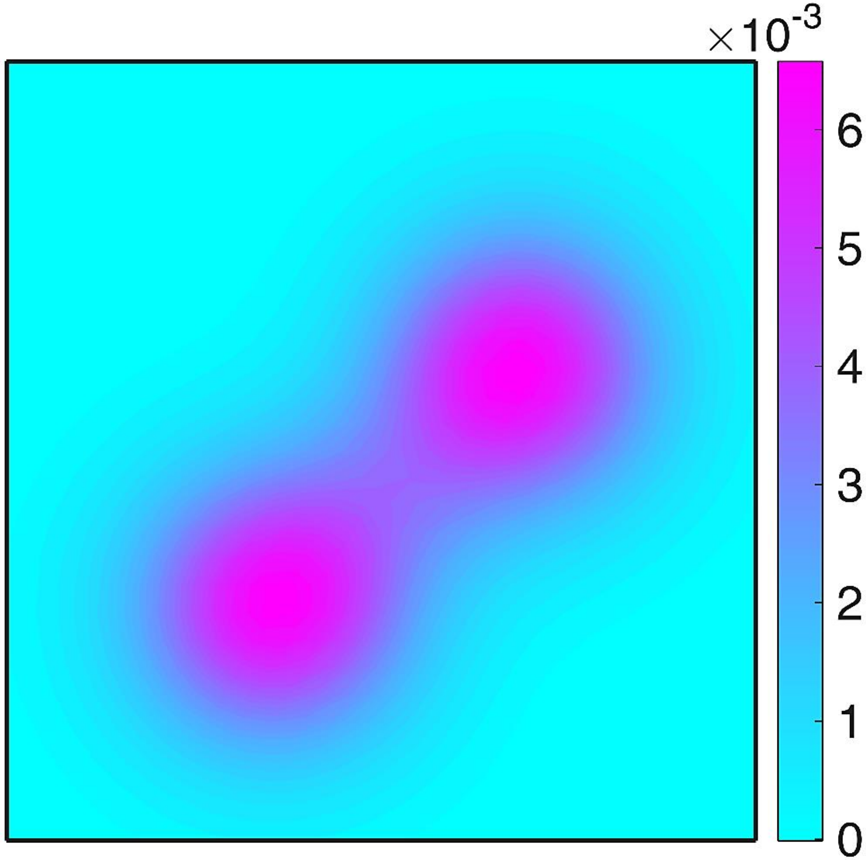


Bi-merons

Spin configuration

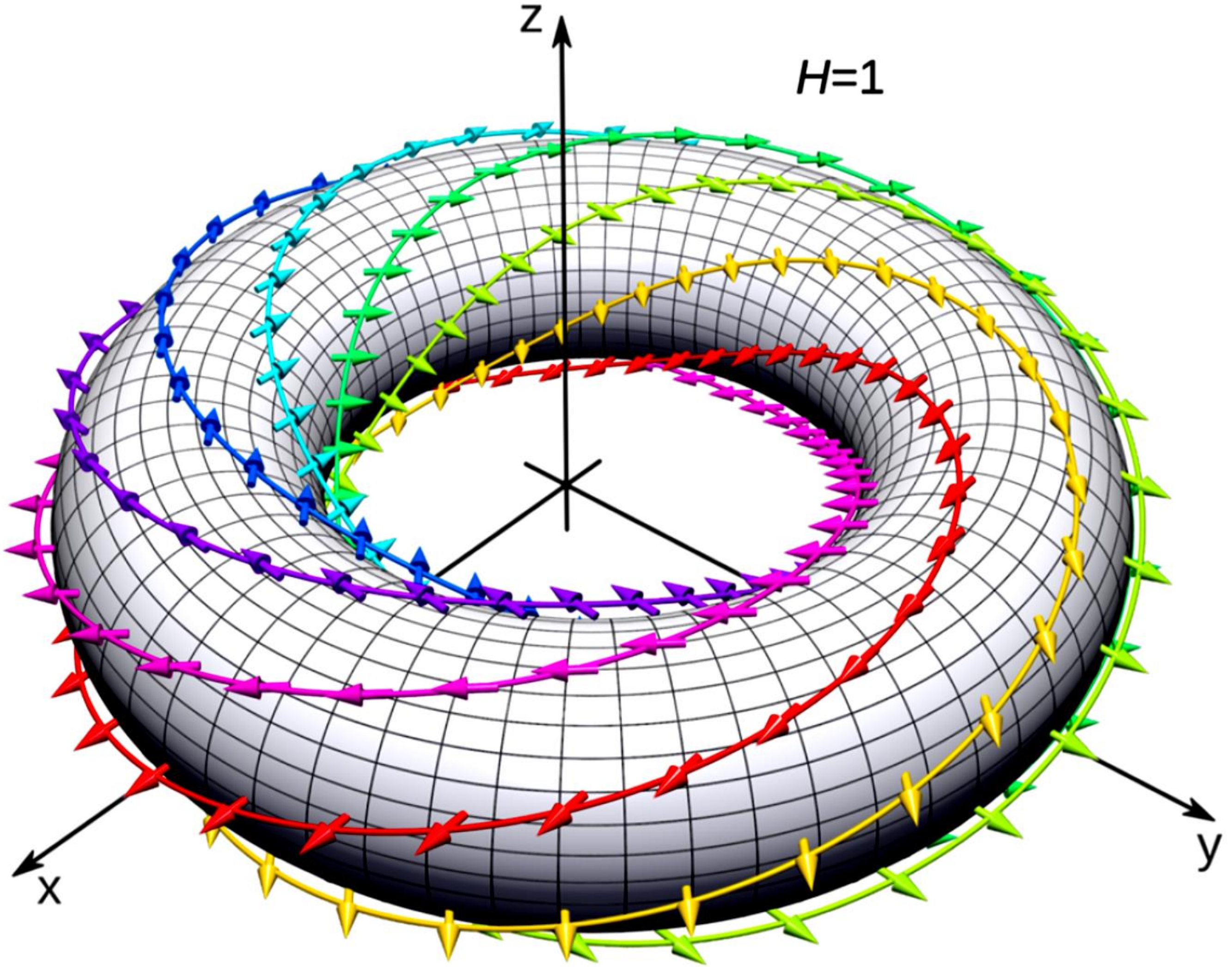


Topological density



MM in "The 2020 skyrmionics roadmap", J. Phys. D: Appl. Phys. 53, 363001 (2020)

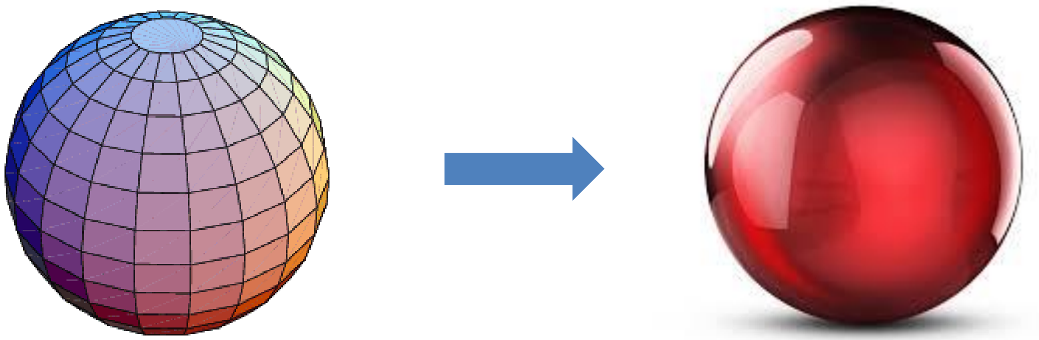
Hopfion



F. N. Rybakov et al. APL Mater. 10, 111113 (2022)

Topological numbers

Gauss-Bonnet theorem

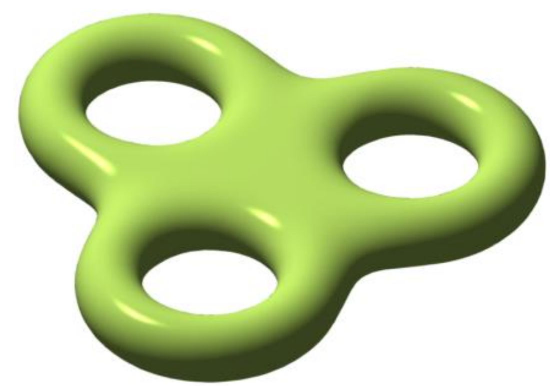


$$\chi = 2$$

$$\chi = \frac{1}{2\pi}\int dS\, K = 2 - 2h$$

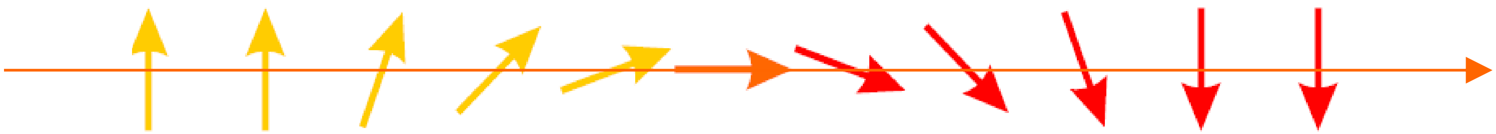
$$\chi = 0$$

$$\chi = -2$$



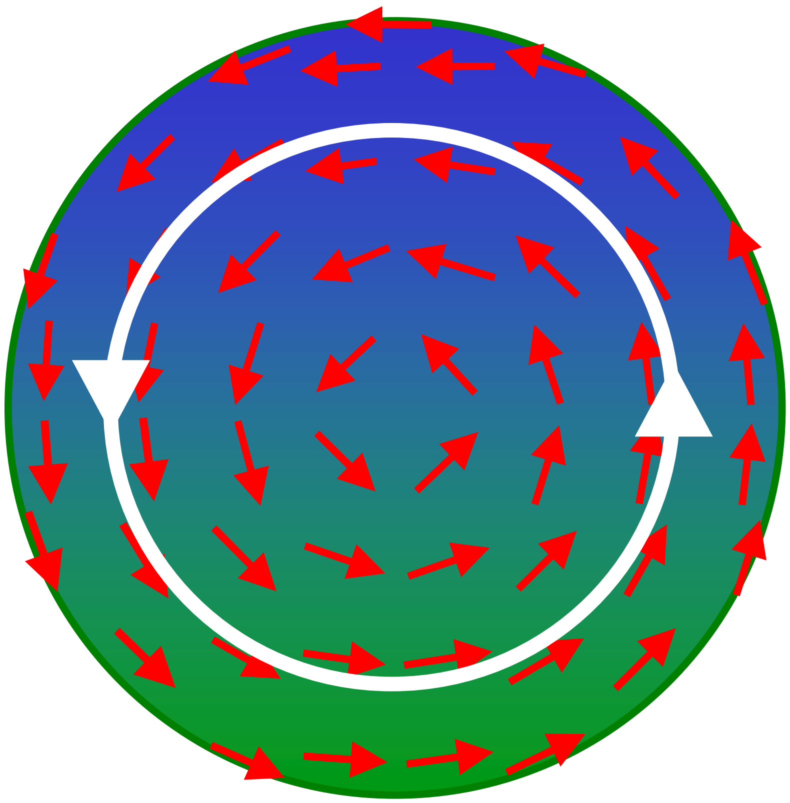
$$\chi = -4$$

Magnetic domain wall



$$Q = \frac{1}{\pi}\int dx \frac{d\phi}{dx} = \frac{\phi(+\infty) - \phi(-\infty)}{\pi} = \pm 1$$

Topological charge of magnetic vortex



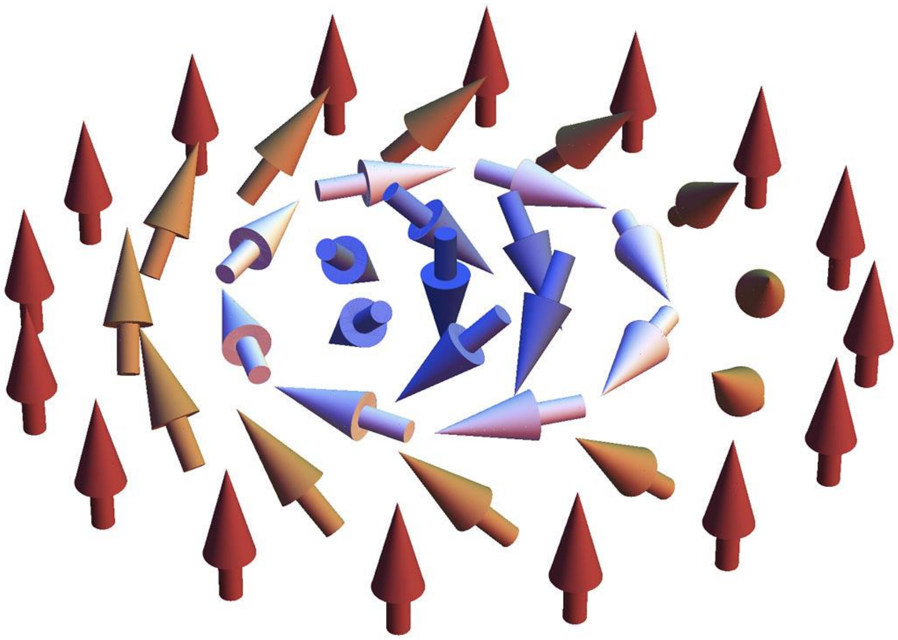
$$\mathbf{n} = \hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \sin\phi$$

Winding number

$$Q = \frac{1}{2\pi} \oint d\mathbf{x} \cdot \nabla \phi$$

$$S^1 \to S^1$$

2D Skyrmion

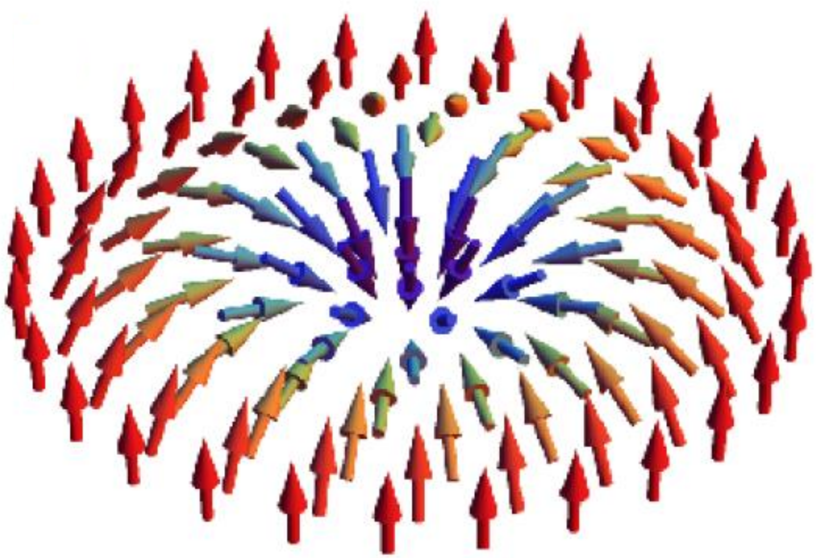


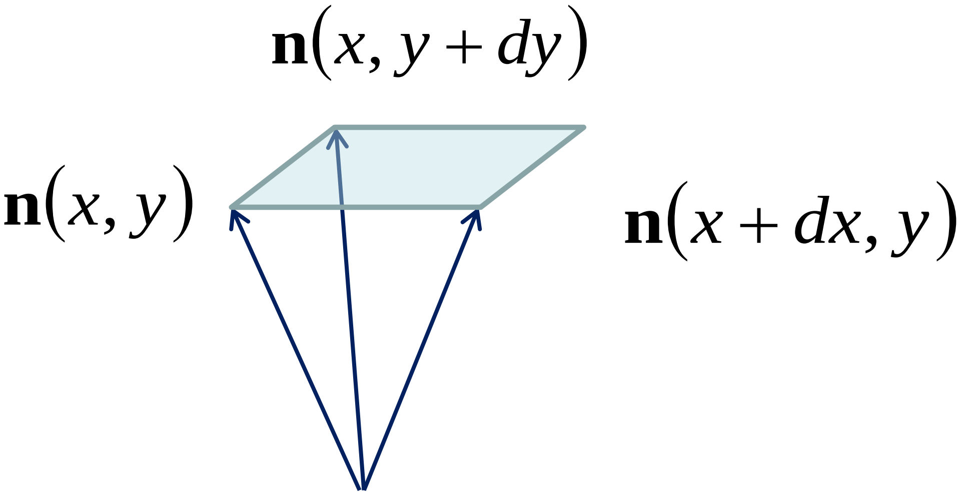
$$n_x^2 + n_y^2 + n_z^2 = 1$$

$$S^2 \to S^2$$

Topological charge

$$Q = \frac{1}{4\pi}\int d^2x\left(\mathbf{n}\cdot\left[\partial_x \mathbf{n}\times\partial_y \mathbf{n}\right]\right) = \frac{1}{4\pi}\int d\Omega = \text{integer } \#$$





#1.1 Skyrmion topological current

$$j^{\alpha}=(\rho, j^{x}, j^{y})=\frac{1}{8\pi}\varepsilon^{\alpha\beta\gamma}\left(\mathbf{n}\cdot\partial_{\beta}\mathbf{n}\times\partial_{\gamma}\mathbf{n}\right)$$

$$x^{\alpha}=(t,\ x, y) \quad n_x^2+n_y^2+n_z^2=1 \qquad \varepsilon^{012}=+1$$

$$\text{Prove that } \partial_{\alpha}j^{\alpha}=\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{j}=0$$

Hints

$$\left(\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}\right)=0, \text{if the three vectors are co-planar}$$

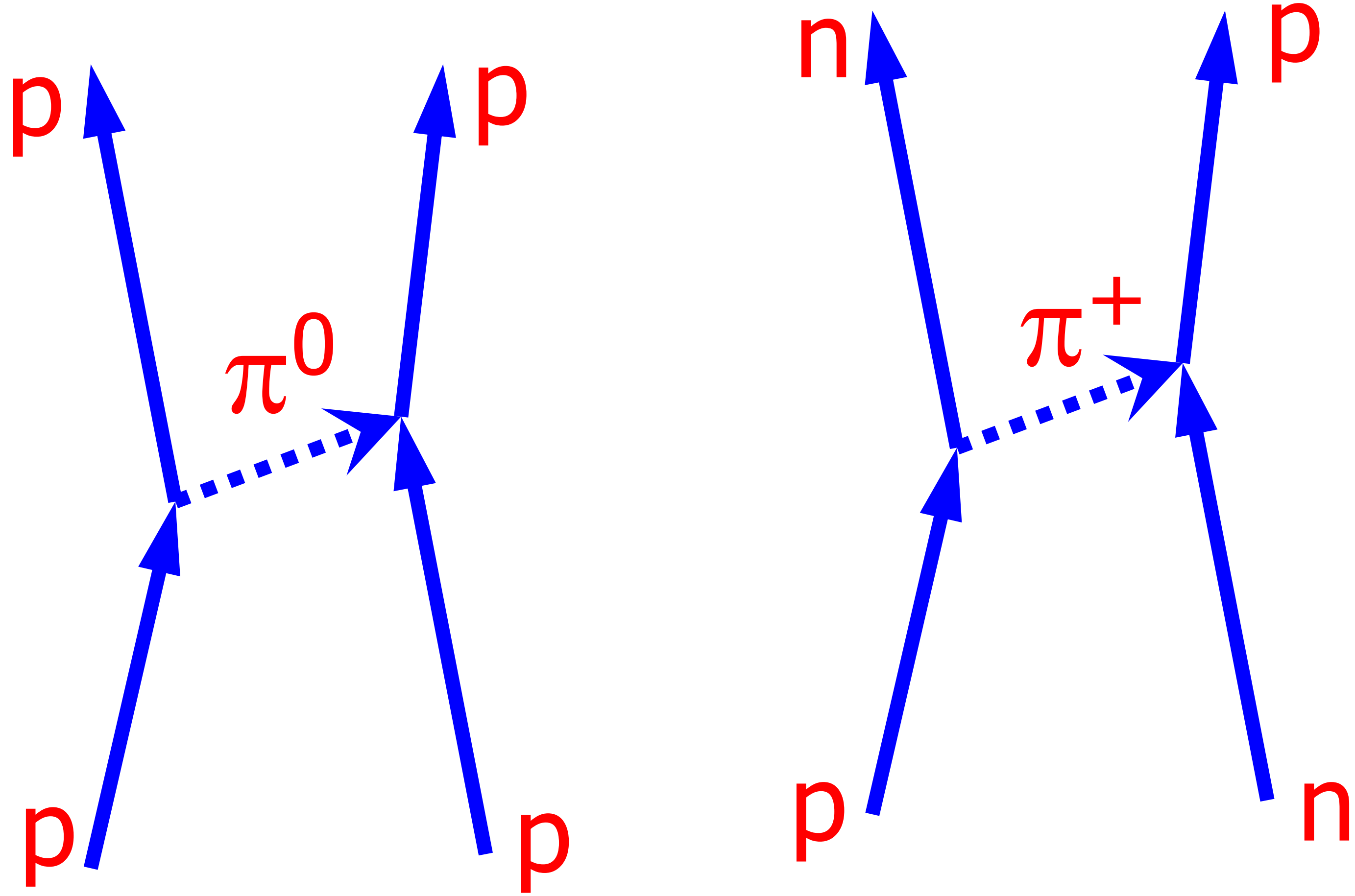
$$\left(\mathbf{n}\cdot\partial_{\alpha}\mathbf{n}\right)=0$$

Magnetic particles

Nucleons and pions

$$U(x) \propto e^{-\frac{x}{l}}$$

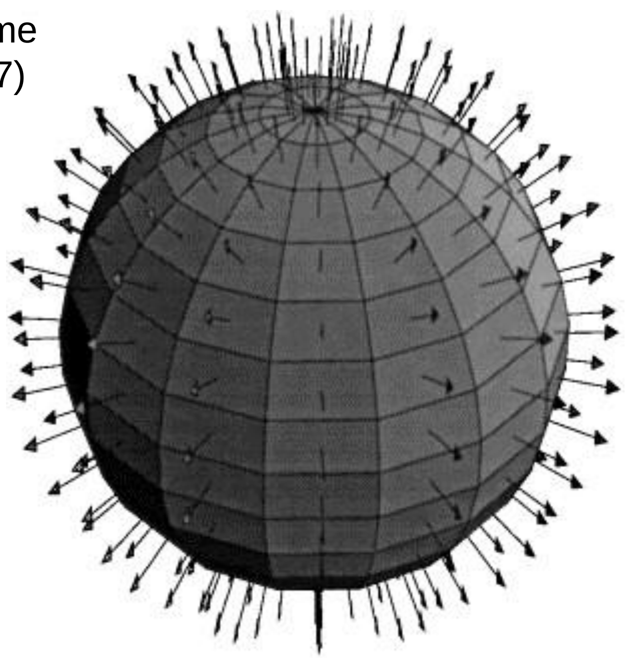
$$l = \frac{\hbar}{m_\pi c}$$



Skyrme's Skyrmion

T. H. R. Skyrme, A Nonlinear field theory, Proc. Roy. Soc. London ***A 260****, 127 (1961)*

Tony Skyrme
(1922-1987)



hedgehog

$$\pi_x^2 + \pi_y^2 + \pi_z^2 + \sigma^2 = const$$

Topological soliton

$$S^3 \to S^3$$

$$\pi_3(S^3) = \mathbb{Z}$$

Topological charge = baryon number

Skyrmions are fermions, spin, isospin, baryon properties, NN-interaction, πN-scattering, SkX = nuclear matter

Topological charge

Coordinates on S³ $$\phi_0^2 + \phi_x^2 + \phi_y^2 + \phi_z^2 = 1$$

Solid angle $$d\Omega = \varepsilon_{\alpha\beta\gamma\delta}\phi_\alpha \partial_x \phi_\beta \partial_y \phi_\gamma \partial_z \phi_\delta \, d^3x$$

Area of S³ $$A = 2\pi^2$$

Topological charge

$$Q = \frac{1}{A}\int d\Omega = \frac{1}{2\pi^2}\int d^3x \varepsilon_{\alpha\beta\gamma\delta}\phi_\alpha \partial_x \phi_\beta \partial_y \phi_\gamma \partial_z \phi_\delta$$

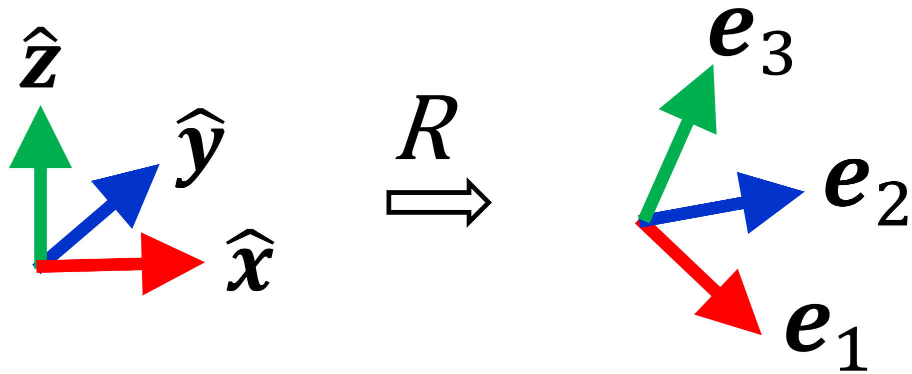
From S^3 to real world

S^3 $\rightarrow$ SU(2) $\rightarrow$ SO(3)

$$U = \phi_0 + i\boldsymbol{\phi}\cdot\boldsymbol{\sigma} \in \mathrm{SU}(2), \quad U^\dagger U = 1, \det(U) = 1$$

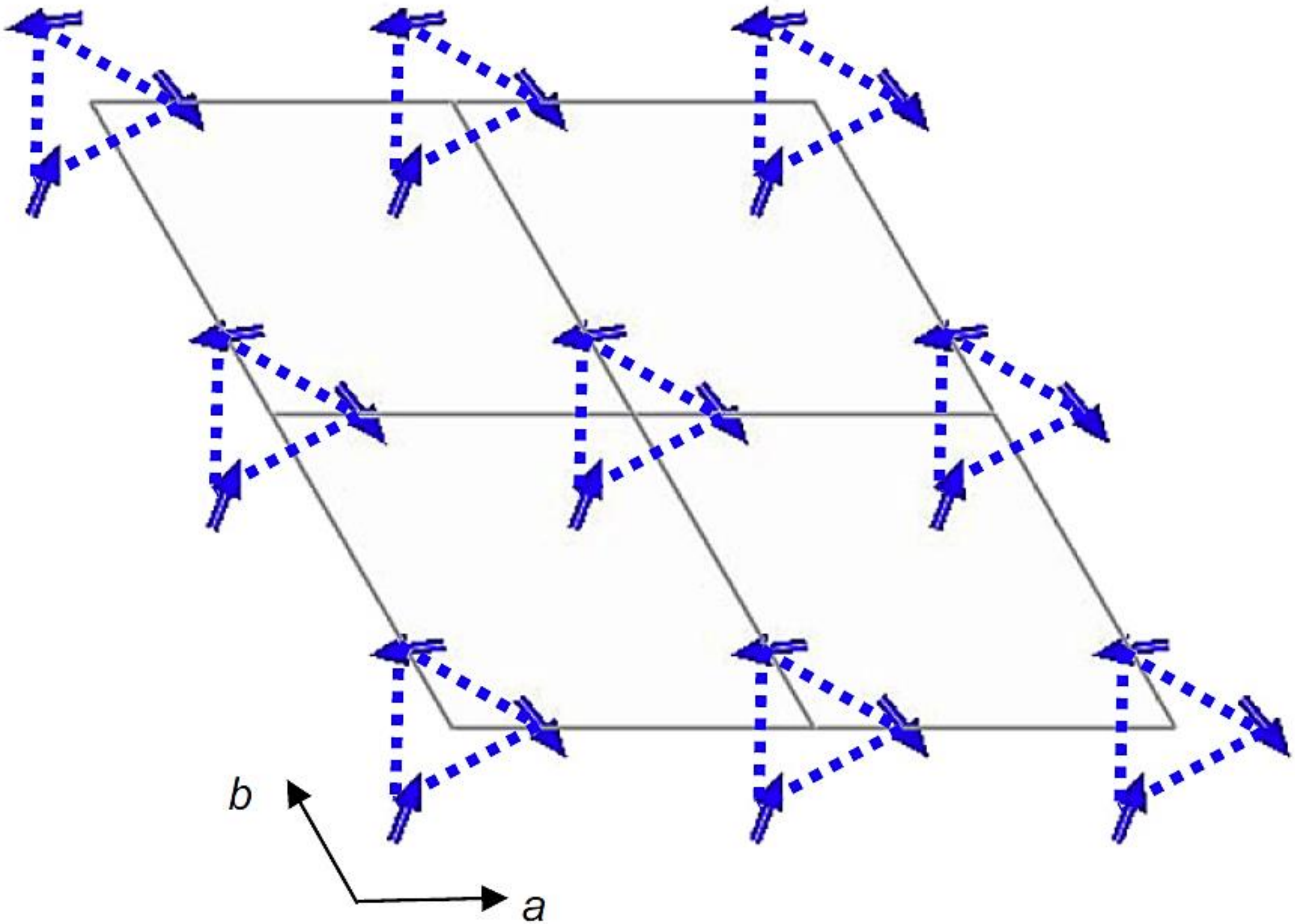
$$R_{ab} = \frac{1}{2}\mathrm{tr}\left(\sigma_a U^\dagger \sigma_b U\right) \in \mathrm{SO}(3) \qquad \pm U \rightarrow R$$

SO(3) is S^3 with identified antipodal points



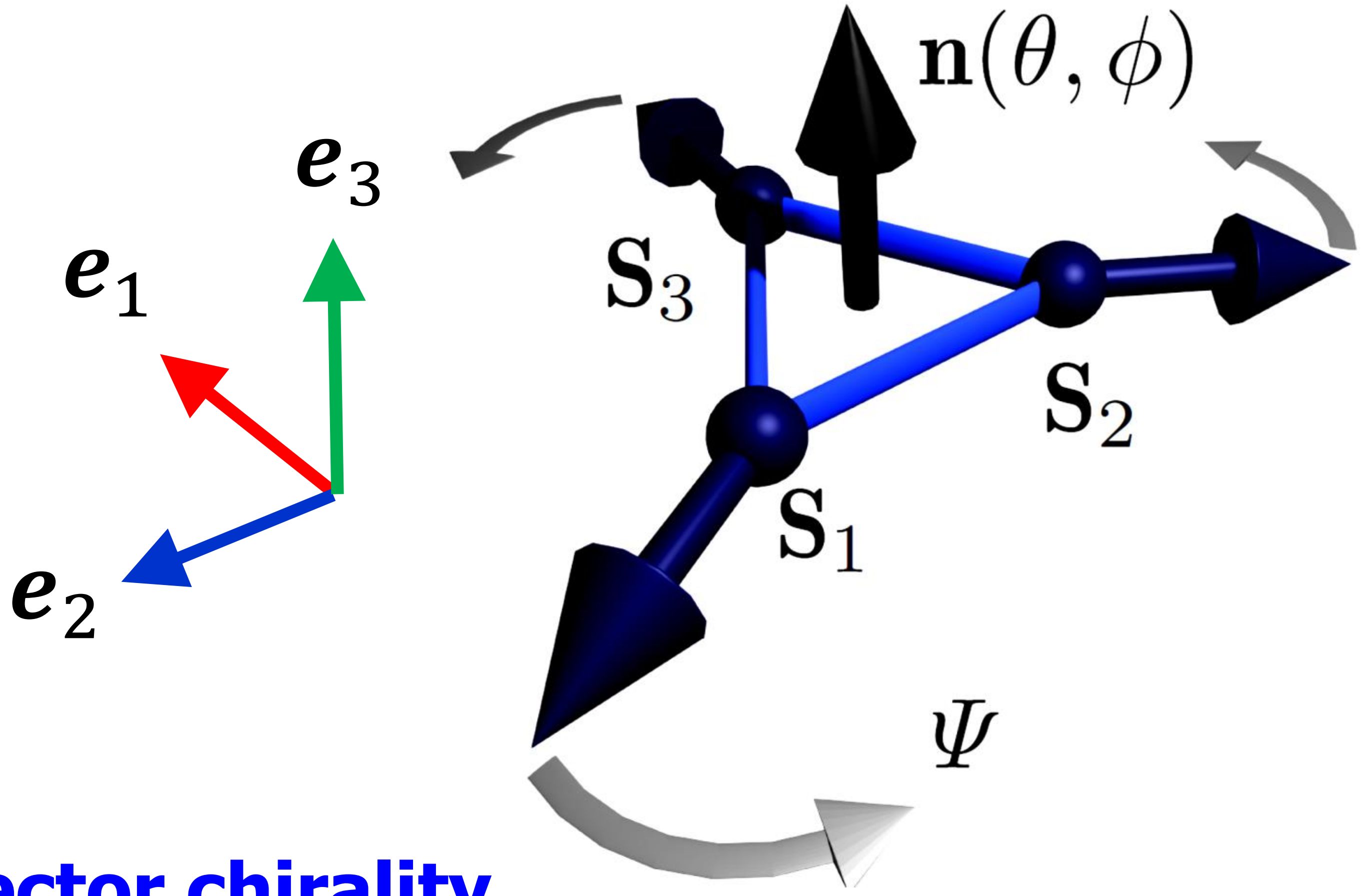
Physical systems with SO(3) order parameter: superfluid ^{3}He, non-collinear antiferromagnets

Fe-langasite $Ba_3TaFe_3Si_2O_{14}$



K. Marty et al., PRL 101, 247201 (2008)

SO(3) order parameter

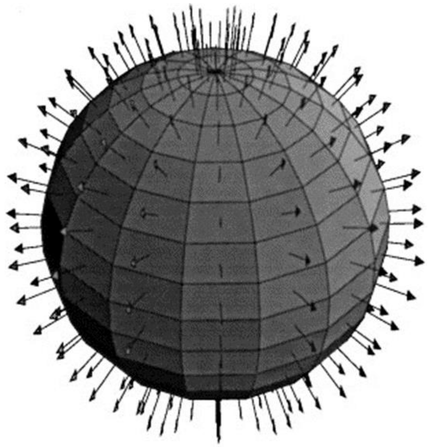


Vector chirality

$$\mathbf{n} = \frac{2}{3\sqrt{3}}\left(\mathbf{S}_1 \times \mathbf{S}_2 + \mathbf{S}_2 \times \mathbf{S}_3 + \mathbf{S}_3 \times \mathbf{S}_1\right)$$

3D Skyrmion → Shankar monopole

3D Skyrmion



$$\phi_0 = \cos\frac{\alpha(r)}{2} \qquad \boldsymbol{\phi} = \hat{\boldsymbol{r}}\sin\frac{\alpha(r)}{2}$$

$$\alpha(0) = 2\pi \qquad \alpha(\infty) = 0$$

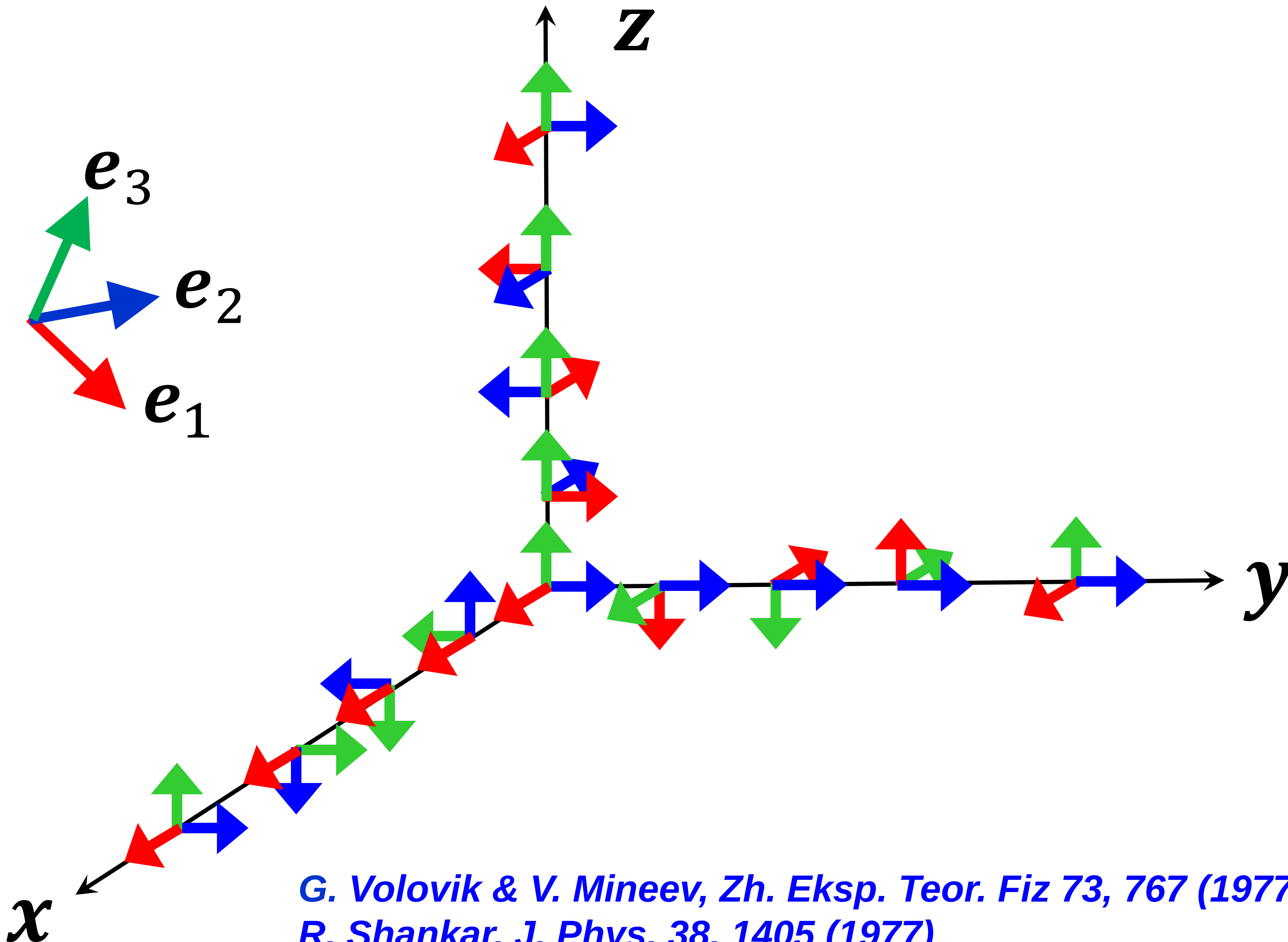
$$U = \phi_0 + i\boldsymbol{\phi}\cdot\boldsymbol{\sigma} = e^{\frac{i}{2}\alpha(r)\hat{\boldsymbol{r}}\cdot\boldsymbol{\sigma}} \in \mathrm{SU}(2)$$

Shankar monopole

$$R_{ab} = \frac{1}{2}\mathrm{tr}\left(\sigma_a U^\dagger \sigma_b U\right) = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) \in \mathrm{SO}(3)$$

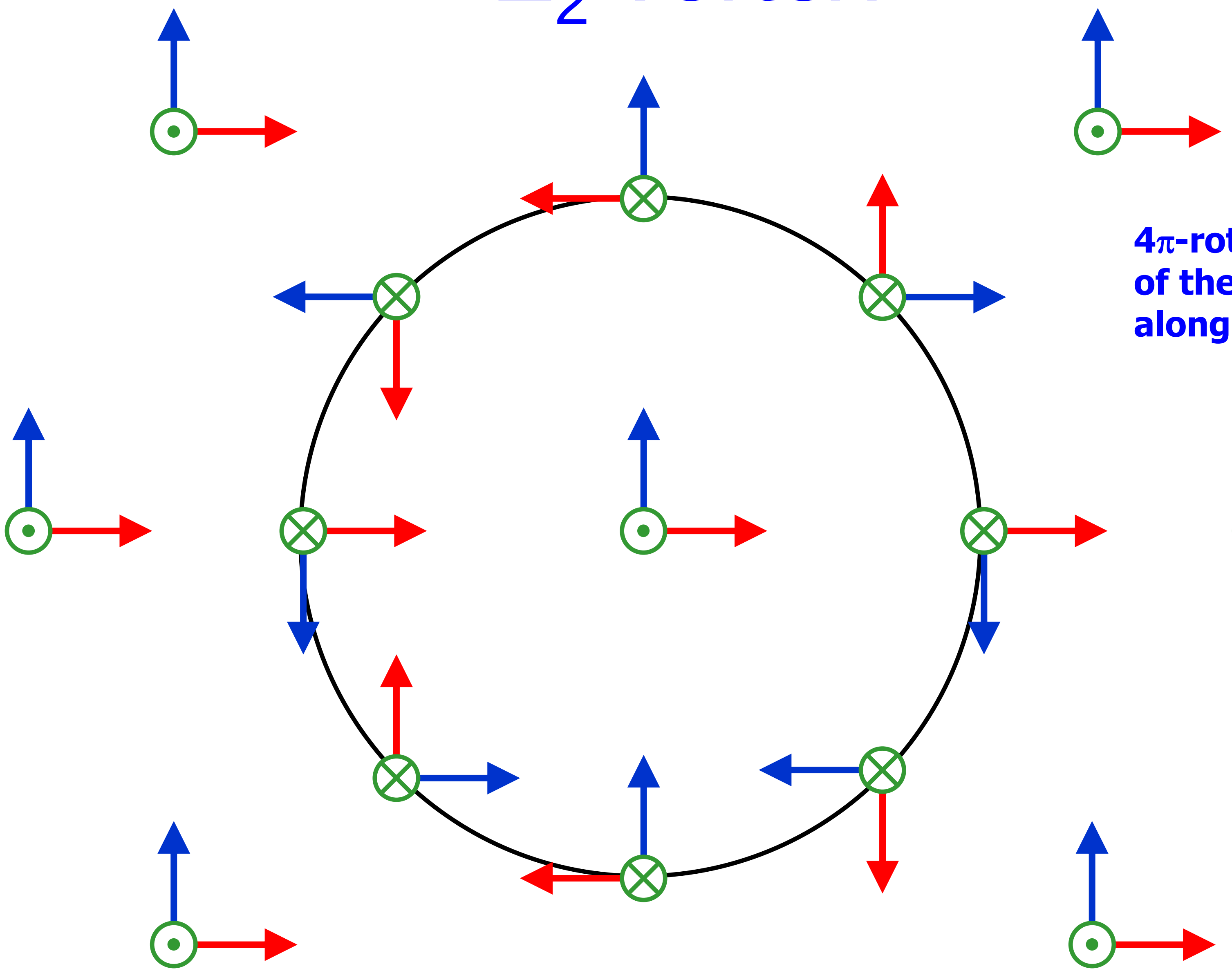
rotation around $\hat{\boldsymbol{r}}$ through the angle α

Shankar monopole



G. Volovik & V. Mineev, Zh. Eksp. Teor. Fiz 73, 767 (1977)
R. Shankar, J. Phys. 38, 1405 (1977)

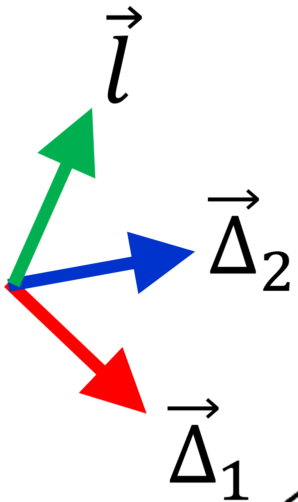
Z_2 vortex

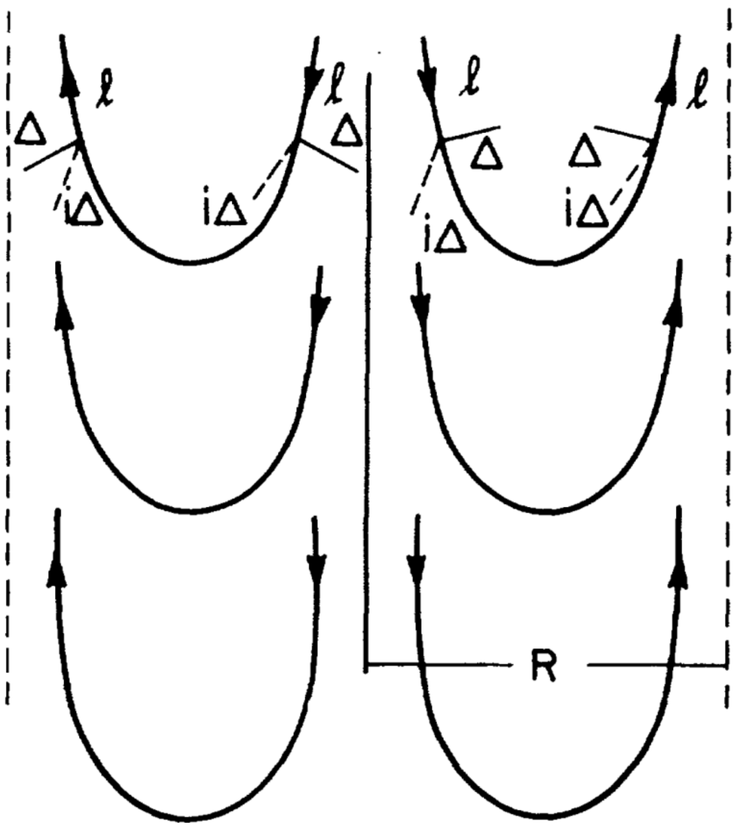


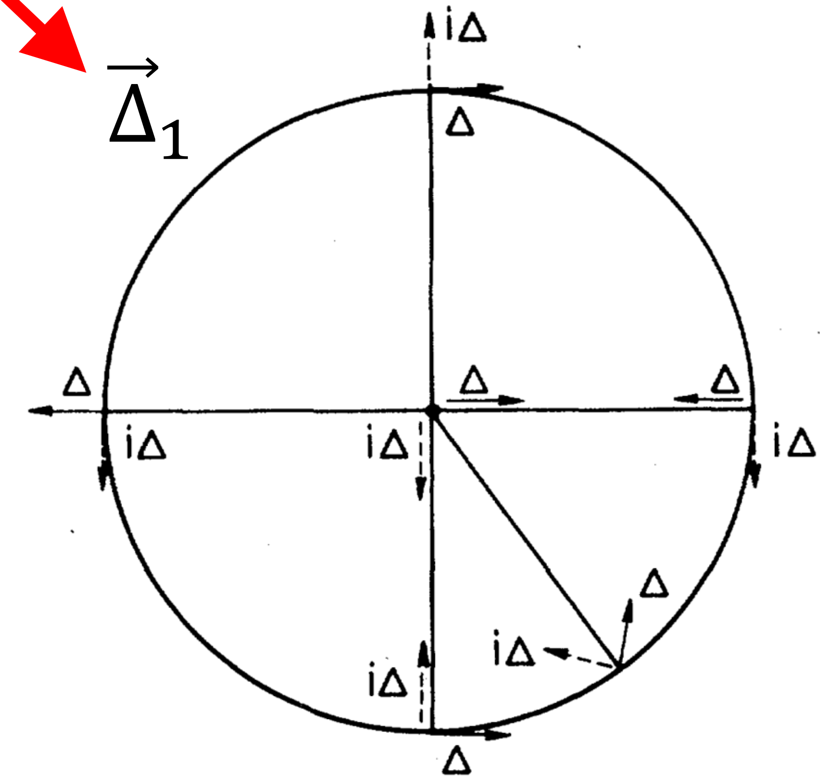
4π-rotation of the frame along the circle

Coreless vortex in superfluid ^{3}He-A

$$\vec{\Delta}(\boldsymbol{x}) = \vec{\Delta}_1(\boldsymbol{x}) + i\,\vec{\Delta}_2(\boldsymbol{x})$$

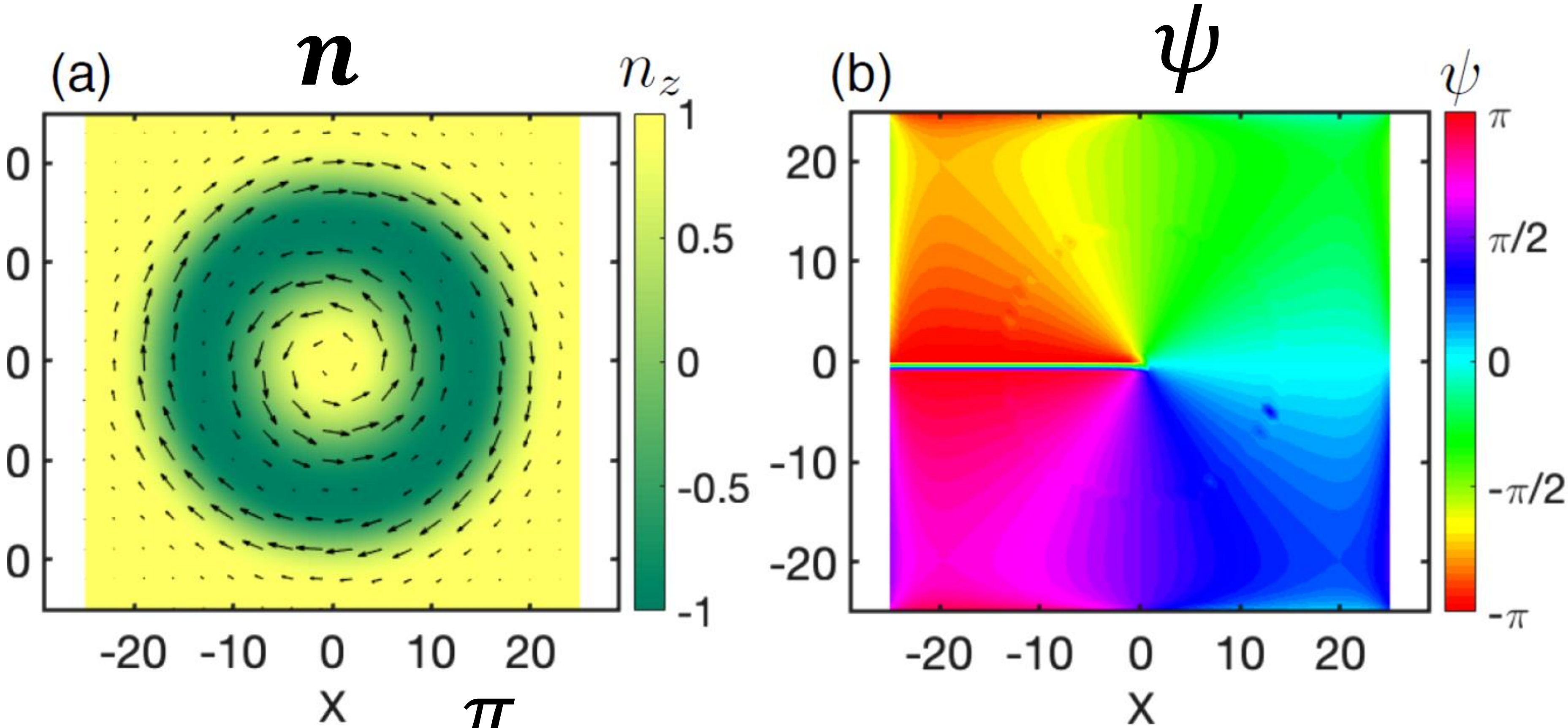






P.W. Anderson & G. Thoulouse, PRL 38, 508 (1977)

Z_2-vortex without core in chiral Fe-langasite



$$\phi = \varphi + \frac{\pi}{2} \qquad \psi = -\phi \qquad \theta \in [0,2\pi]$$

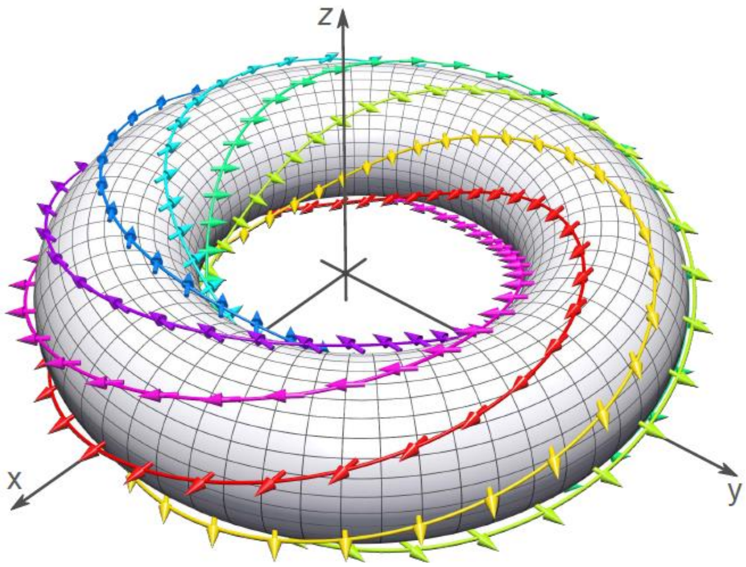
Evgenii Barts & MM, npj Quantum Mater. 6, 104 (2021)

3D Skyrmion → Hopfion

$$R_{ab} = \frac{1}{2}\mathrm{tr}\left(\sigma_a U^\dagger \sigma_b U\right) = (\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) \in \mathrm{SO}(3)$$

Hopfion $\boldsymbol{n} = \boldsymbol{e}_3(\boldsymbol{r})$

$$\begin{cases} n_x = +\sin\alpha\,\hat{y} + (1-\cos\alpha)\hat{x}\hat{z} \\ n_y = -\sin\alpha\,\hat{x} + (1-\cos\alpha)\hat{y}\hat{z} \\ n_z = \cos\alpha + (1-\cos\alpha)\hat{z}\hat{z} \end{cases}$$



Emergent electrodynamics

Adiabatic motion of spin-polarized electrons

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\triangle - J\boldsymbol{\sigma}\cdot\boldsymbol{n}\right)\psi \qquad \boldsymbol{n} = \boldsymbol{n}(\boldsymbol{x}, t)$$

Adiabatic projection

$$|\boldsymbol{n}\rangle = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} \\ \sin\frac{\theta}{2}e^{+i\phi/2} \end{pmatrix}$$

$$|\psi\rangle = \chi|\boldsymbol{n}\rangle \qquad \boldsymbol{\sigma}\cdot\boldsymbol{n}|\boldsymbol{n}\rangle = |\boldsymbol{n}\rangle$$

$$i\hbar\frac{\partial\chi}{\partial t} = \left(-ea_0 + \frac{1}{2m}\left(\frac{\hbar}{i}\boldsymbol{\nabla} + \frac{e}{c}\boldsymbol{a}\right)^2 - J + \frac{\hbar^2}{8m}\partial_i\boldsymbol{n}\cdot\partial_i\boldsymbol{n}\right)\chi$$

Effective gauge potentials

$$a_0 = \frac{i\hbar}{e}\langle\boldsymbol{n}|\frac{\partial}{\partial t}|\boldsymbol{n}\rangle \qquad \boldsymbol{a} = -\frac{i\hbar c}{e}\langle\boldsymbol{n}|\boldsymbol{\nabla}|\boldsymbol{n}\rangle$$

Effective gauge fields

$$\begin{cases} e_i = -\partial_i a_0 - \dfrac{1}{c}\partial_t a_i = \dfrac{\hbar}{2e}\left(\mathbf{n}\cdot\left[\partial_i \mathbf{n}\times\partial_t \mathbf{n}\right]\right) \\ h_i = \left[\nabla\times\mathbf{a}\right]_i = \dfrac{\hbar c}{4e}\varepsilon_{ijk}\left(\mathbf{n}\cdot\left[\partial_j \mathbf{n}\times\partial_k \mathbf{n}\right]\right) \end{cases}$$

#1.2

Lorentz force acting on spin-polarized electron

$$\mathbf{F} = -e\,\mathbf{e} - \frac{e}{c}[\mathbf{v}\times\mathbf{h}]$$

Problem 1.2

$$|\boldsymbol{n}\rangle = \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi/2} \\ \sin\frac{\theta}{2} e^{+i\phi/2} \end{pmatrix}$$

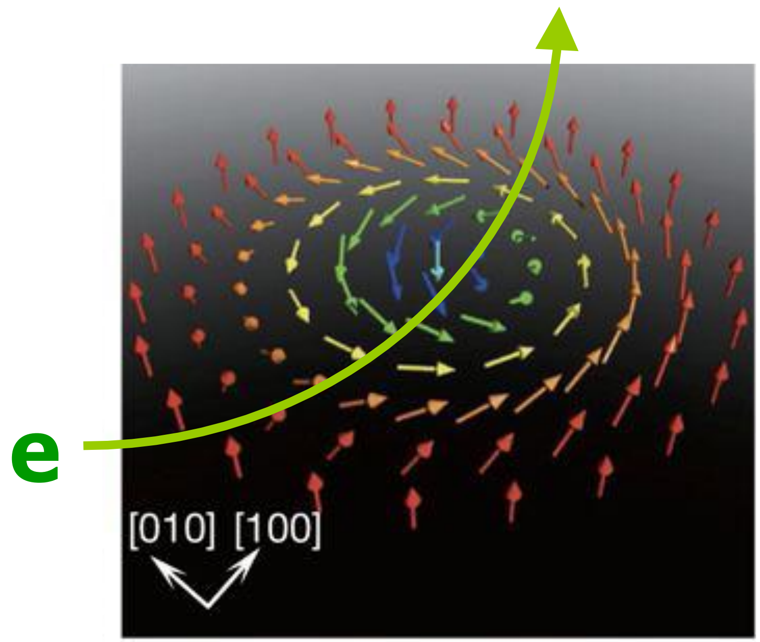
Find expressions for $a_0, \boldsymbol{a}, \boldsymbol{e}, \boldsymbol{h}$

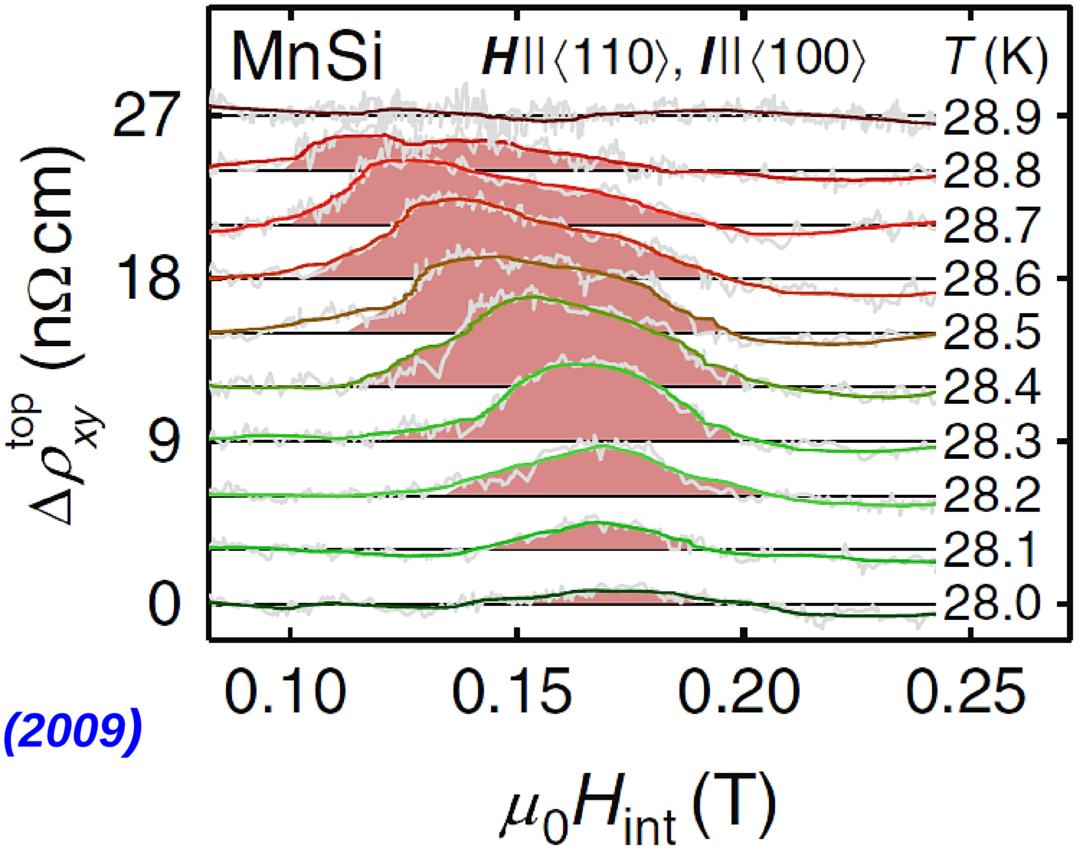
$$\partial_i \boldsymbol{n} = \widehat{\boldsymbol{\theta}} \partial_i \theta + \widehat{\boldsymbol{\phi}} \sin\theta \, \partial_i \phi$$

$$\widehat{\boldsymbol{\theta}} = \frac{\partial \boldsymbol{n}}{\partial \theta} \qquad \widehat{\boldsymbol{\phi}} = \frac{1}{\sin\theta} \frac{\partial \boldsymbol{n}}{\partial \phi}$$

$\boldsymbol{n}, \widehat{\boldsymbol{\theta}}, \widehat{\boldsymbol{\phi}}$ form orthonormal basis $\boldsymbol{n} = \widehat{\boldsymbol{\theta}} \times \widehat{\boldsymbol{\phi}}$

Topological Hall Effect





A. Neubauer et al PRL 102, 186602 (2009)

M. Lee et al PRL 102, 186601 (2009)

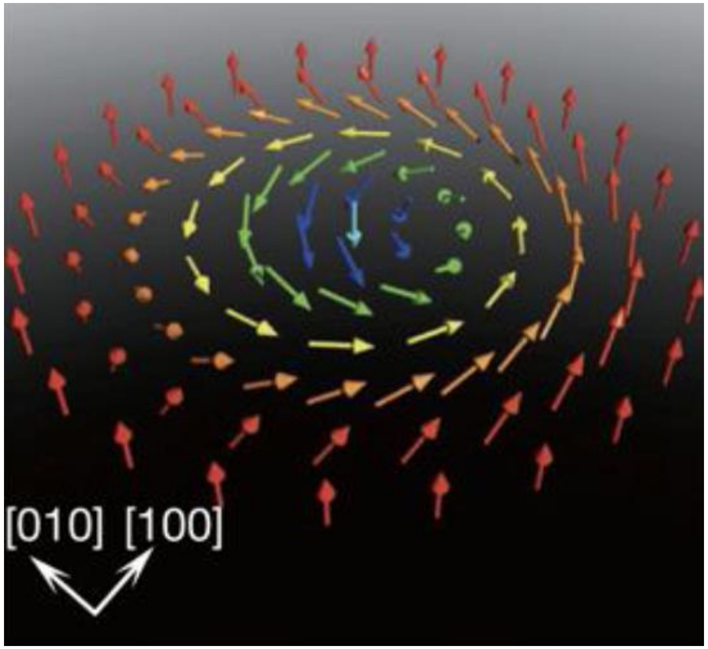
Electric field of moving skyrmion

$$\mathbf{E} = -\frac{1}{c}[\mathbf{V} \times \mathbf{H}]$$

J. Zang et al PRL (2011)

T. Schultz et al Nature Physics (2012)

Physical meaning of Skyrmion charge



Topological charge:

$$Q = \frac{1}{4\pi}\int d^2x\left(\mathbf{n}\cdot\left[\partial_x\mathbf{n}\times\partial_y\mathbf{n}\right]\right)$$

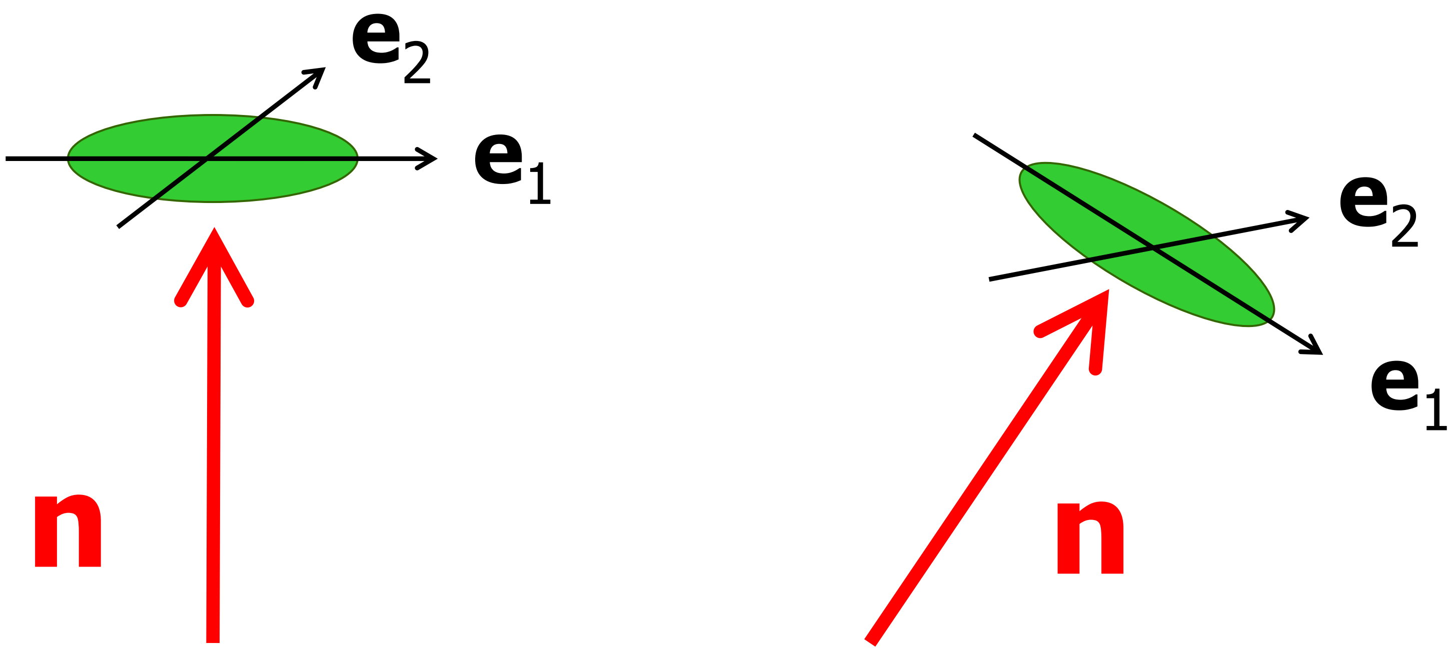
$$Q = \frac{e}{hc}\int d^2x h_z$$

Flux of internal magnetic field:

$$\Phi = \int d^2x h_z = Q\Phi_0 \qquad \Phi_0 = \frac{hc}{e}$$

Magnon effective gauge field

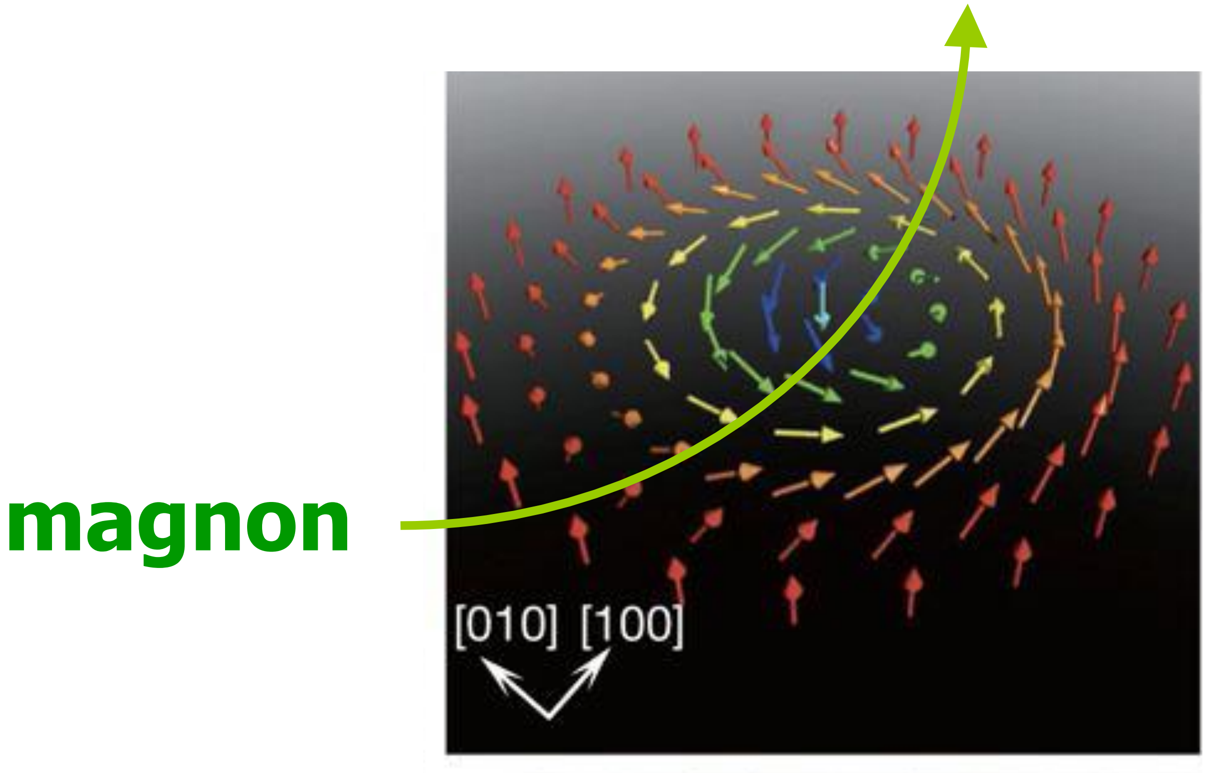
$$\delta\,\mathbf{n}(\mathbf{x},t) = \psi_1 \mathbf{e}_1 + \psi_2 \mathbf{e}_2 \qquad \mathbf{e}_1, \mathbf{e}_2 \perp \mathbf{n}$$



Effective gauge potential

$$a_i = \left(\mathbf{e}_2 \cdot \partial_i \mathbf{e}_1\right)$$

Topological Magnon Hall effect



For magnons, skyrmion is a flux of effective magnetic field

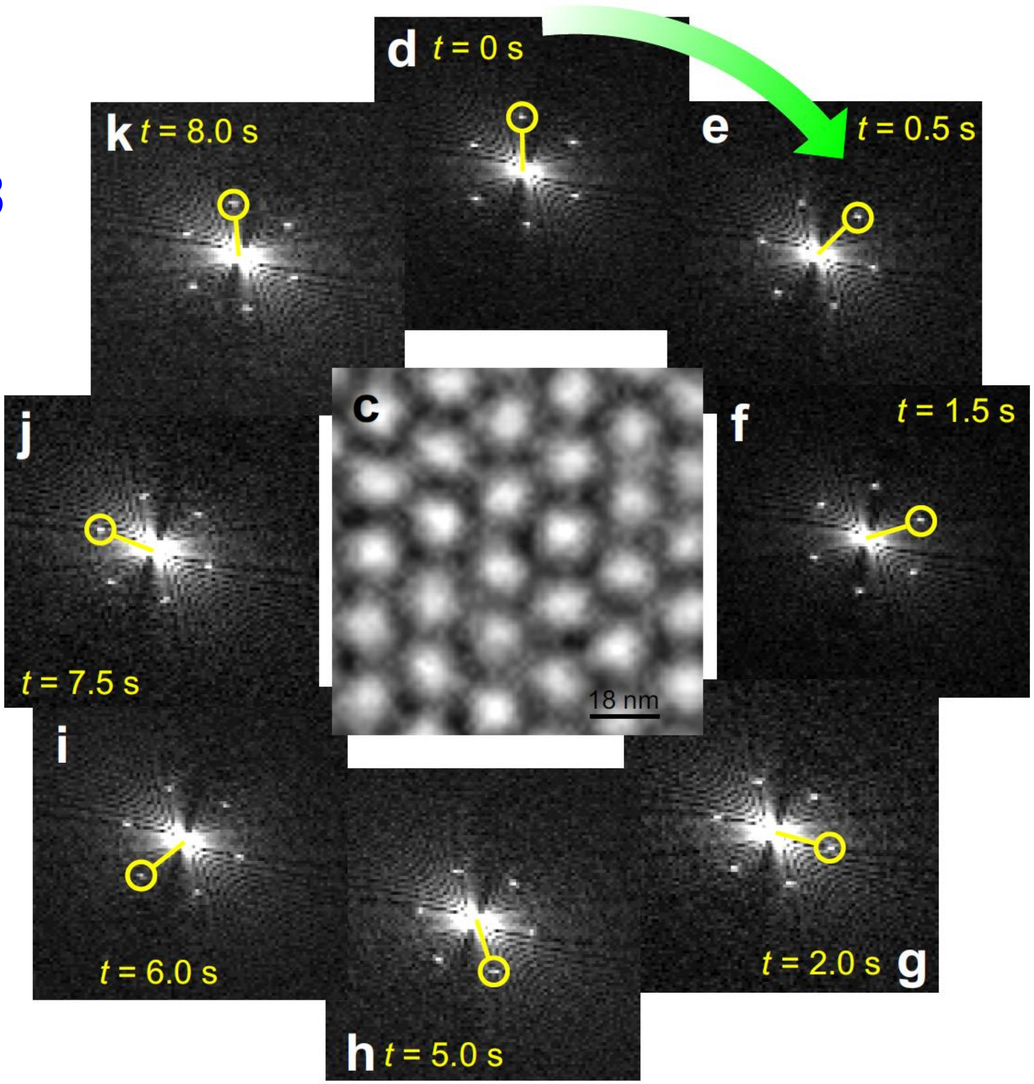
$$\Phi = \pm 2\Phi_0$$

Rotation of skyrmion crystal

M. Mochizuki et al Nature Materials 13, 241 (2014)

Cu_2OSeO_3

MnSi

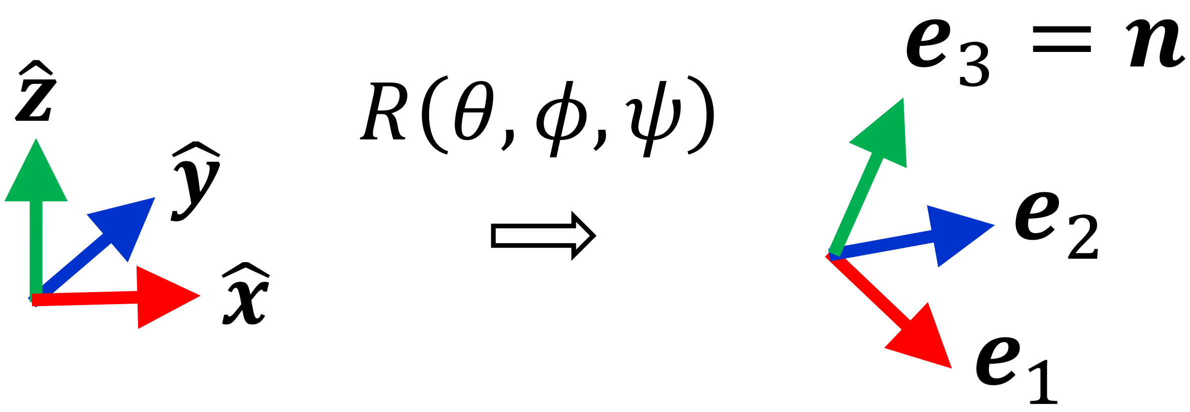


Hopf number and topological charge

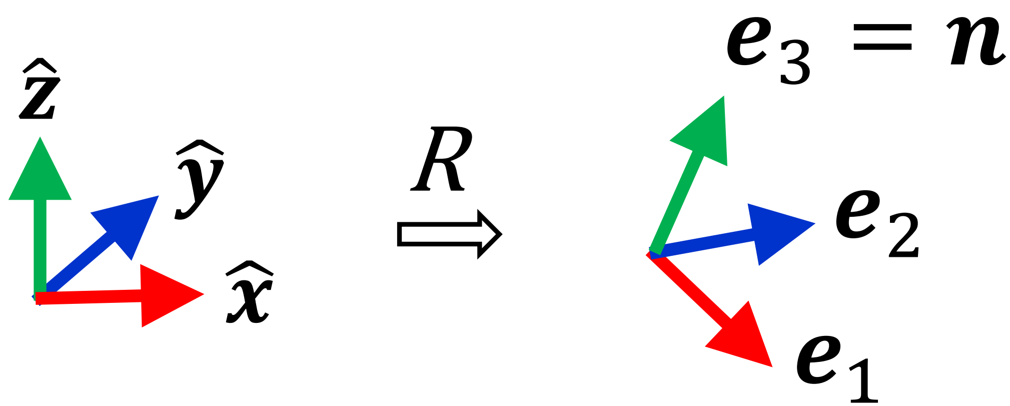
3D skyrmion

$$Q = \frac{1}{12\pi^2}\int d^3x\, \varepsilon_{\alpha\beta\gamma\delta}\varepsilon_{ijk}\phi_\alpha\partial_i\phi_\beta\partial_j\phi_\gamma\,\partial_k\phi_\delta$$

$$= \frac{1}{16\pi^2}\int d^3x\varepsilon_{ijk}\sin\theta\partial_i\,\theta\partial_j\phi\partial_k\psi$$



Hopf number and topological charge



Vector potential and effective magnetic field

$$a_k = -i\langle \boldsymbol{n}|\partial_k|\boldsymbol{n}\rangle = -\frac{1}{2}\boldsymbol{e}_2 \cdot \partial_k \boldsymbol{e}_1 \qquad \boldsymbol{b} = [\boldsymbol{\nabla} \times \boldsymbol{a}]$$

Hopf number (linking number)

$$H = -\frac{1}{4\pi^2}\int d^3x\,(\boldsymbol{a}\cdot\boldsymbol{b}) = Q_{3D}$$

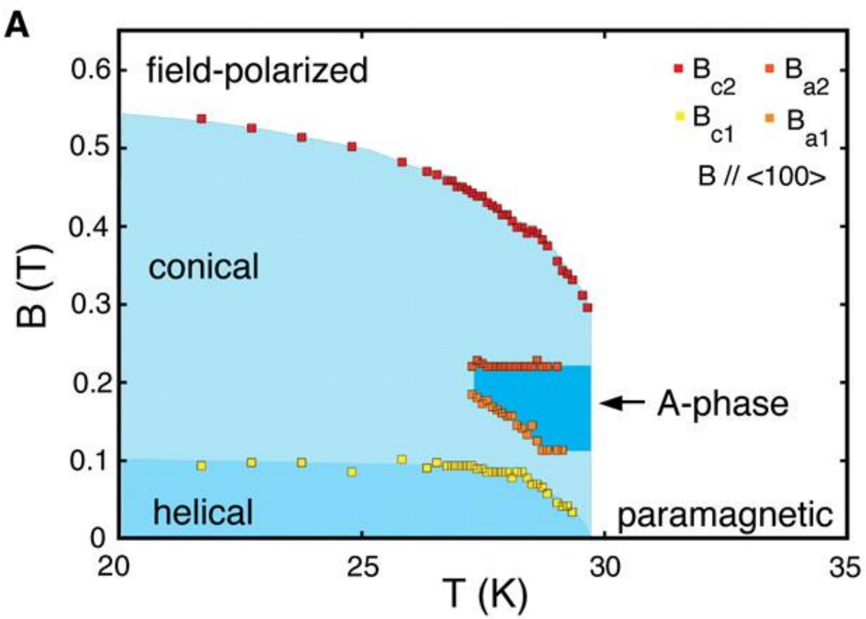
J. H. C. Whitehead, PNAS 33, 117 (1947), A. Kosevich et al. Phys. Rep. 194, 117 (1990)

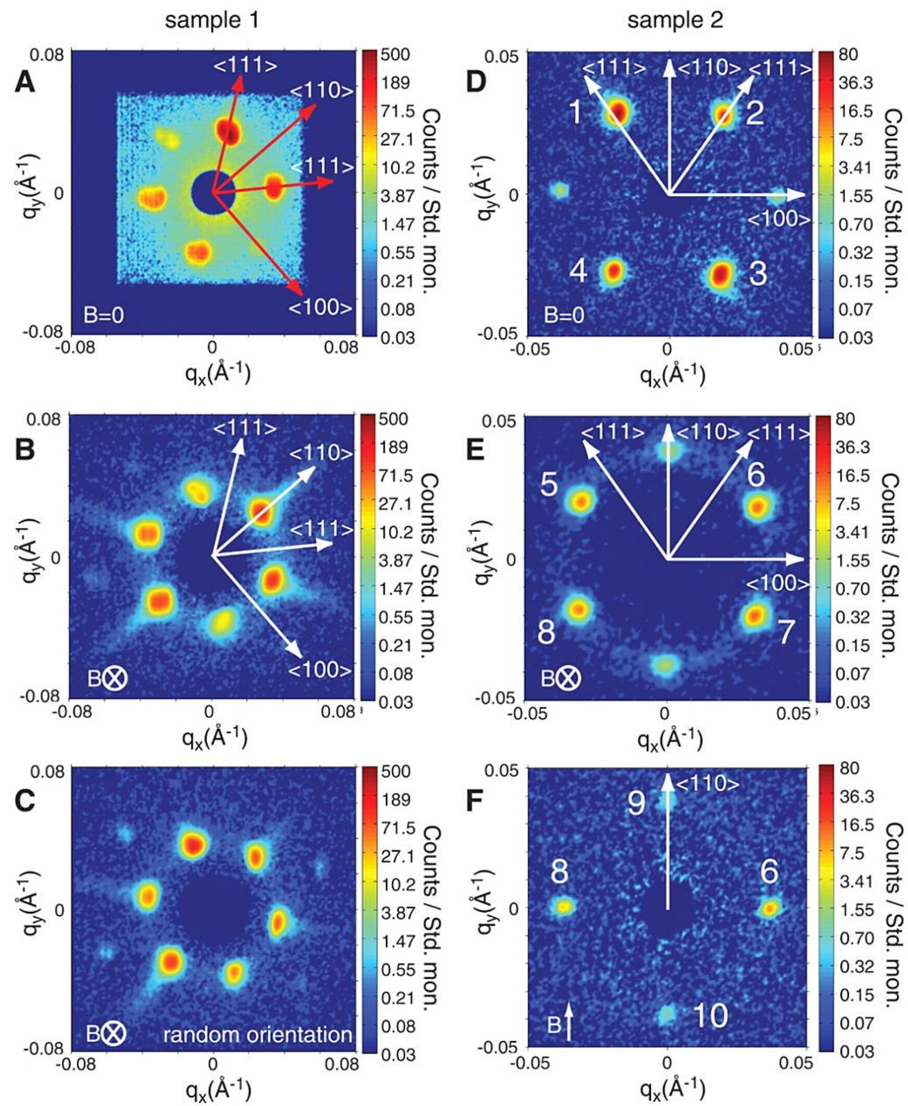
Stability

Small-angle neutron scattering

A.N. Bogdanov & D.A. Yablonskii, Sov. Phys. JETP ***68****, 101 (1989)*

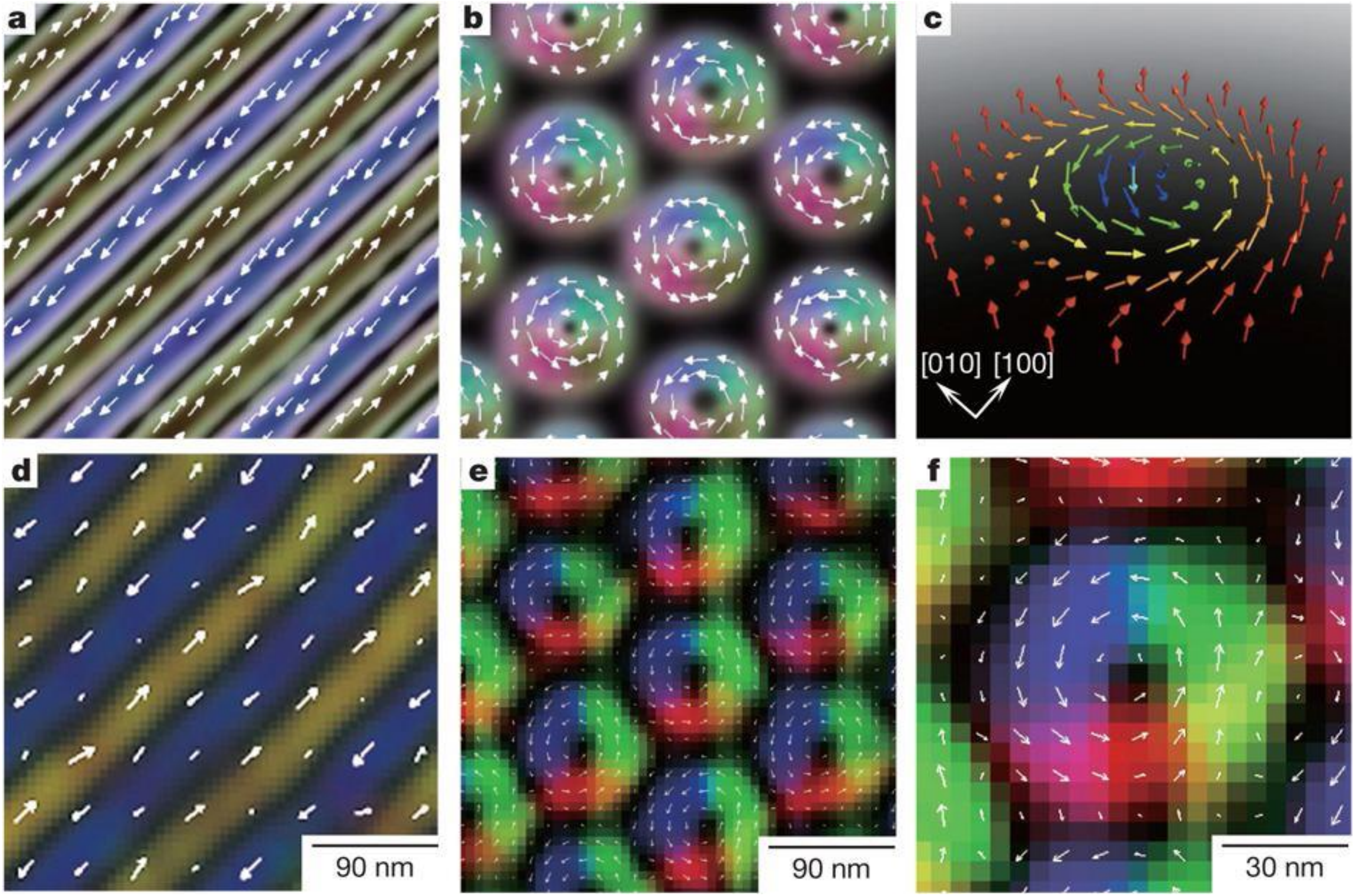
MnSi





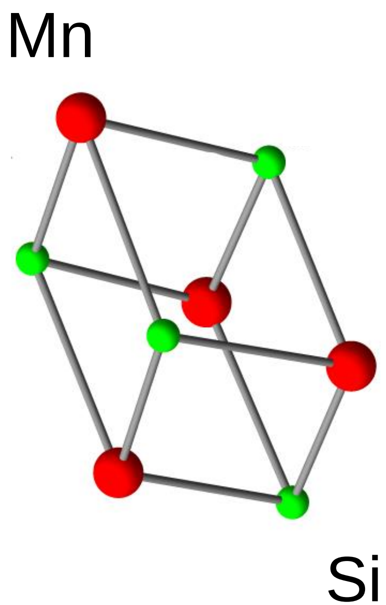
Mühlbauer et al, Science ***323****, 915 (2009)*

Skyrmions imaged by Lorentz microscopy

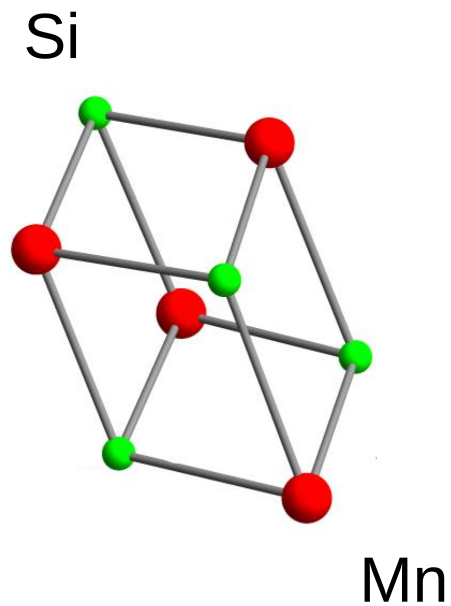


X. Yu et al. (2010)

MnSi



$$\mathbf{x} \rightarrow -\mathbf{x}$$

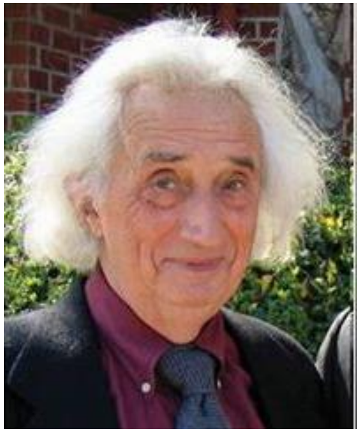


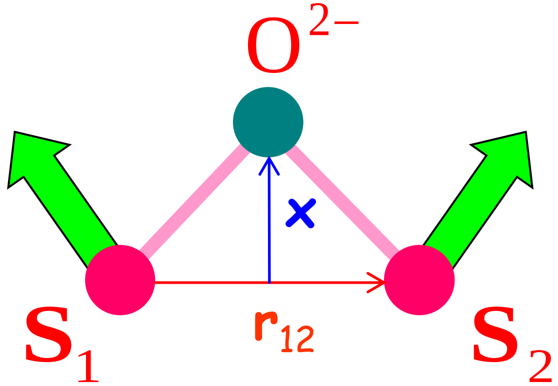
Symmetry class $P2_13$

$$3_{[111]} = (z, x, y) \qquad 2_x, 2_y, 2_z$$

Dzyaloshinskii-Moriya interaction

I. Dzyaloshinskii, Sov. Phys. JETP 19, 960 (1964) T. Moriya, Phys. Rev. 120, 91 (1960)





$$E_{DM} = \mathbf{D}_{12} \cdot [\mathbf{S}_1 \times \mathbf{S}_2]$$

Lifshitz invariants in non-centrosymmetric magnets

$$M_a \partial_b M_c - M_c \partial_b M_a$$

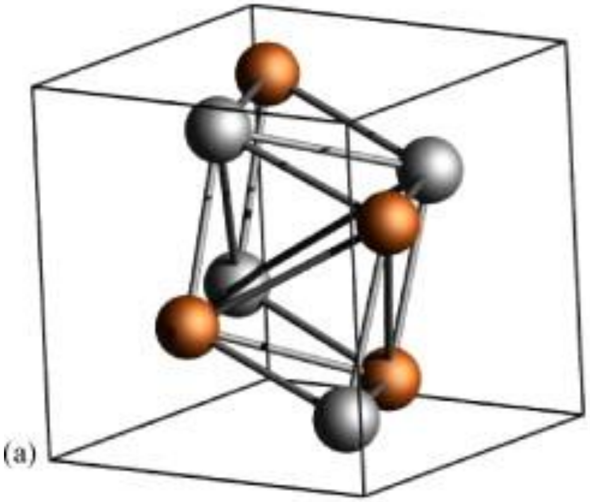
Spin-spiral states

Non-centrosymmetric magnets

MnSi, $Fe_{1-x}Co_xSi$, FeGe
Cu_2OSeO_3

Cubic B20 structure

Class 32



$$f = \frac{J}{2}(\nabla \mathbf{n})^2 + D\,\mathbf{n}\cdot[\nabla \times \mathbf{n}]$$

Exchange energy

Lifshitz invariant

helical spiral wave vector

$$q = D / J$$

#1.3 Lifshitz invariant for MnSi

Symmetry class of MnSi is 32

$$3_{[111]} = (z, x, y)$$

$$2_x = (x, -y, -z) \qquad 2_y, 2_z$$

Show that the following term is allowed by symmetry

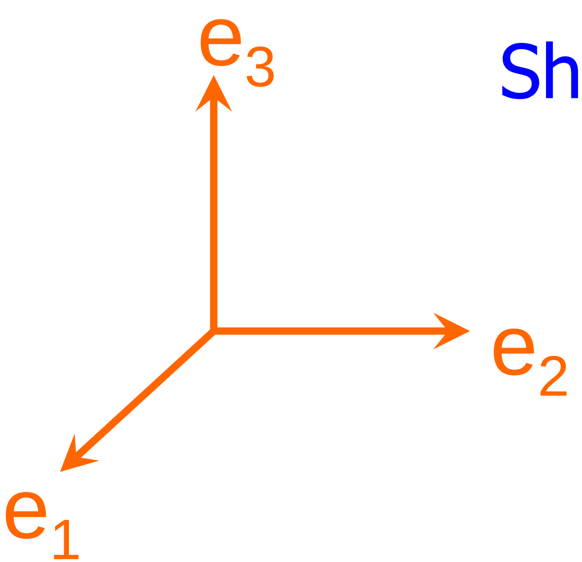
$$\mathbf{n} \cdot [\nabla \times \mathbf{n}] = n_x \partial_y n_z + n_y \partial_z n_x + n_z \partial_x n_y - n_x \partial_z n_y - n_y \partial_x n_z - n_z \partial_y n_x$$

#1.4 Helical spiral in MnSi

Consider a spiral

$$\mathbf{n} = \mathbf{e}_1 \cos(\mathbf{q}\cdot\mathbf{x}+\varphi) + \mathbf{e}_2 \sin(\mathbf{q}\cdot\mathbf{x}+\varphi),$$

$\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$ is the rotation axis



Show that $D\,\mathbf{n}\cdot\left[\nabla\times\mathbf{n}\right]$ is minimal, if

$$\begin{cases} \mathbf{q} \uparrow\uparrow \mathbf{e}_3, \text{ for } D>0 \\ \mathbf{q} \uparrow\downarrow \mathbf{e}_3, \text{ for } D<0 \end{cases}$$

Stability of an isolated skyrmion

$$E = \int d^2x \left[\frac{J}{2} (\nabla \mathbf{n})^2 + D\, \mathbf{n} \cdot [\nabla \times \mathbf{n}] + B(1 - n_z) \right] > 0$$

$E_{ex} > 0$ $E_{DM} < 0$ $E_{Zeeman} > 0$

Scaling: $\mathbf{n}(\mathbf{x}) \to \mathbf{n}(\mathbf{x}/\Lambda)$

$$E_\Lambda = \int d^2x \left[\frac{J}{2} (\nabla \mathbf{n})^2 + \Lambda D\, \mathbf{n} \cdot [\nabla \times \mathbf{n}] + \Lambda^2 B(1 - n_z) \right]$$

R^0 R^1 R^2

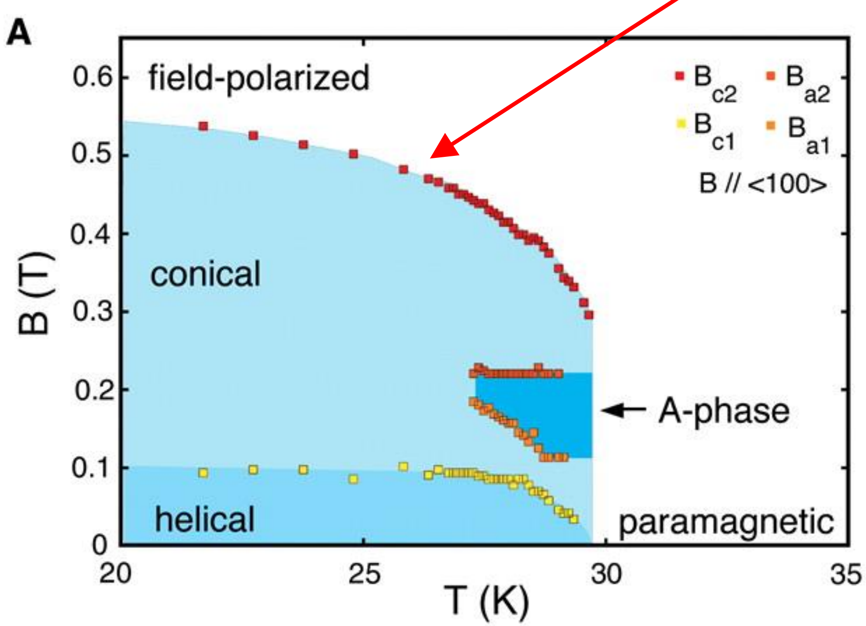
$$\left. \frac{dE_\Lambda}{d\Lambda} \right|_{\Lambda=1} = 0 \qquad E_{\mathrm{DM}} + 2E_{\mathrm{Zeeman}} = 0$$

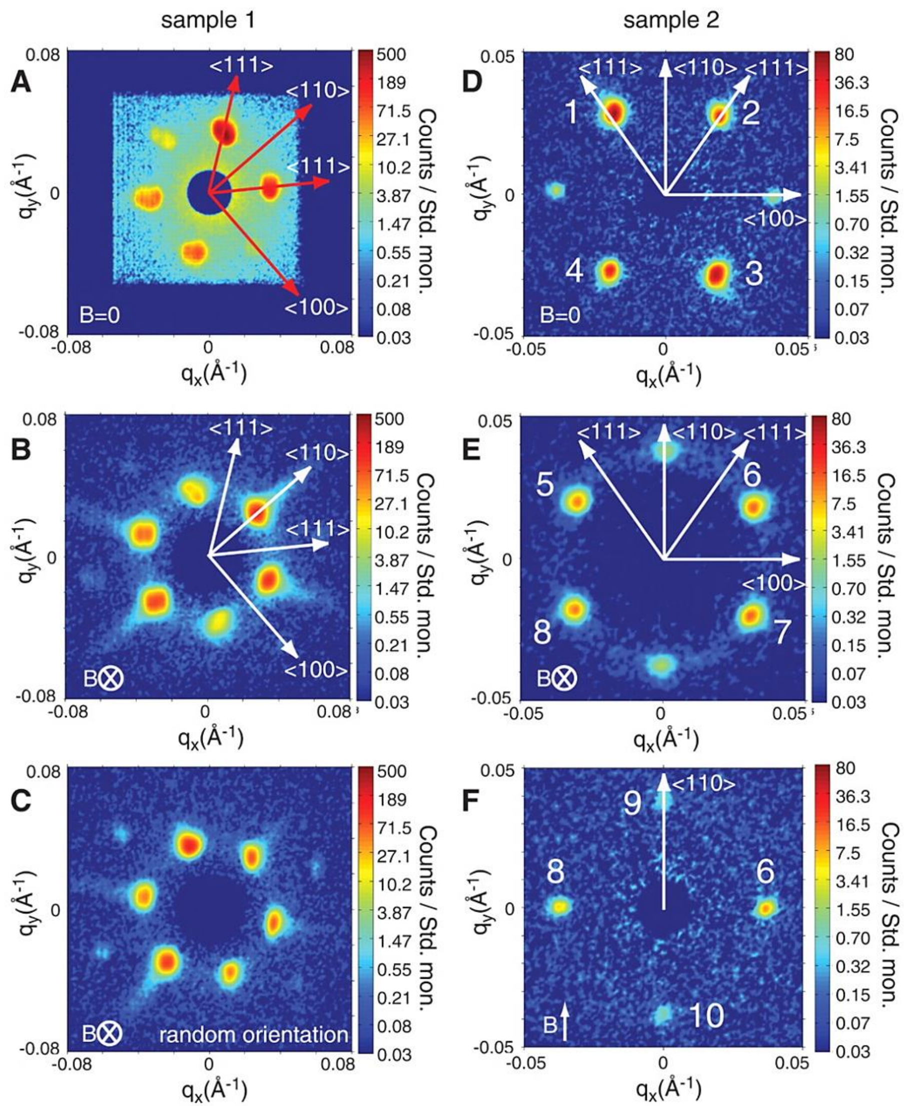
Small-angle neutron scattering

A.N. Bogdanov & D.A. Yablonskii, Sov. Phys. JETP ***68****, 101 (1989)*

isolated skyrmions

MnSi



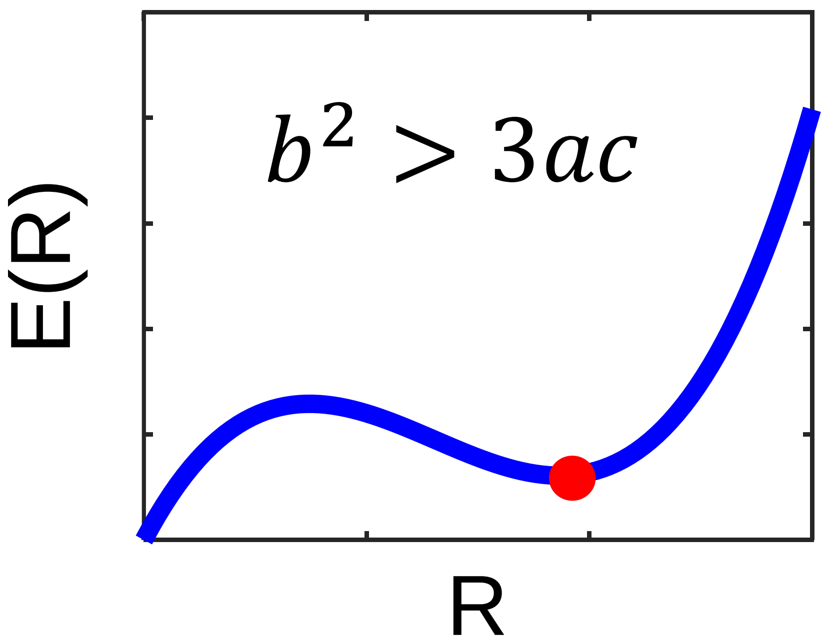


Mühlbauer et al, Science ***323****, 915 (2009)*

Hopfion stability in chiral magnets

Energy $$E = E_{ex} + E_{DM} + E_{ani}$$

Scaling $$E(R) = aR - bR^2 + cR^3$$



$$a \propto J \quad b \propto D \quad c \propto K$$

$$\frac{D^2}{JK} > \lambda_1$$

Stability of uniform FM state

DW energy $E_{DW} = E_{ex} + E_{DM} + E_{ani}$

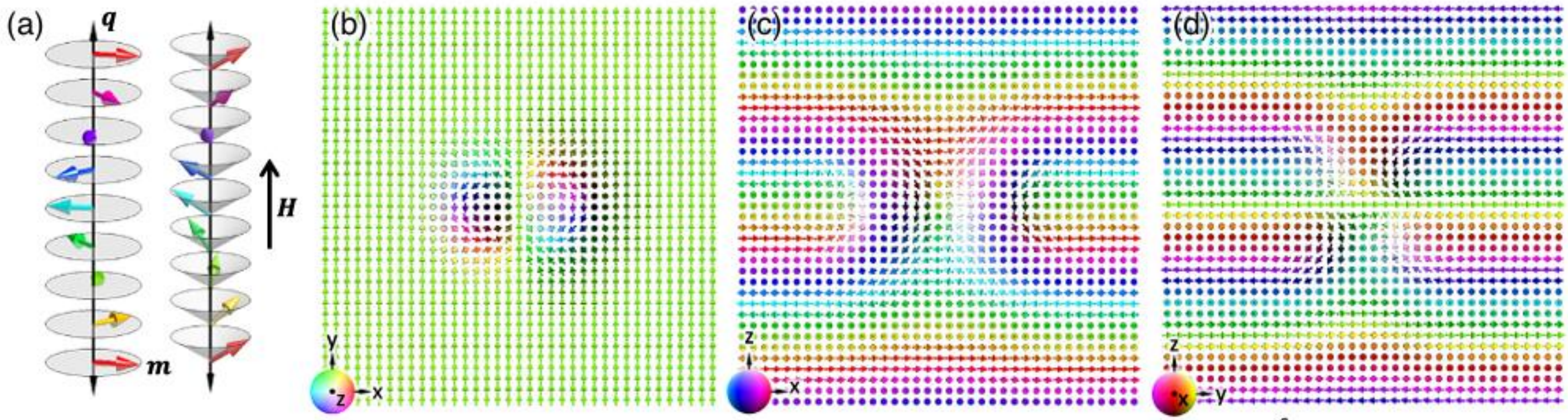
Scaling $E_{DW}(R) = a'R^{-1} - b' + c'R$

DW width $R = \sqrt{\dfrac{a'}{c'}}$

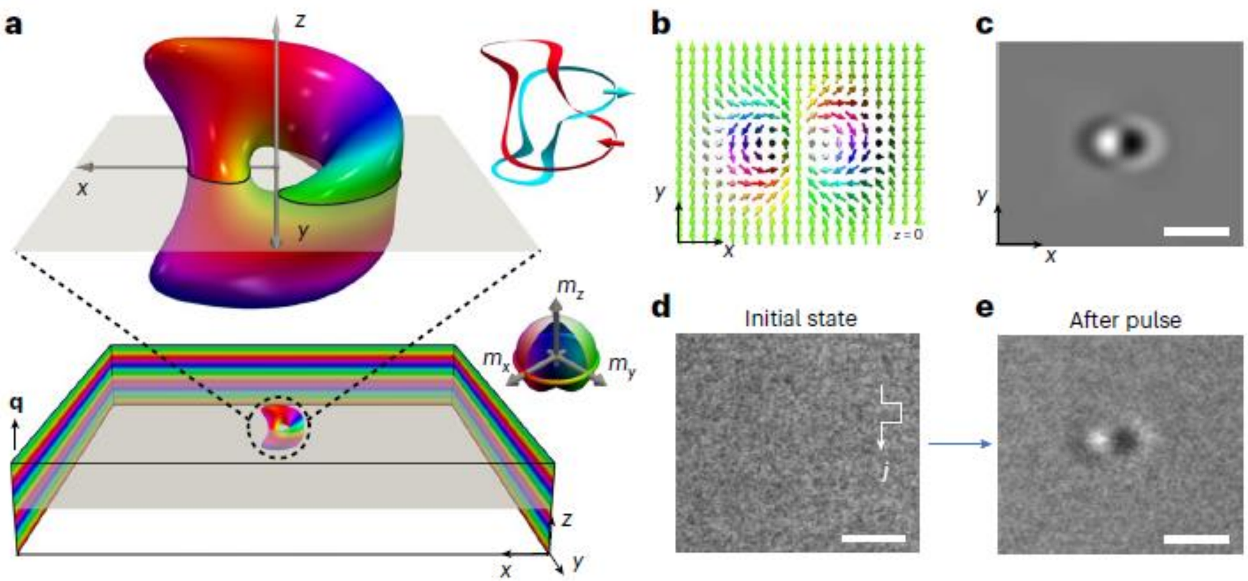
$$E_{DW} = 2\sqrt{a'c'} - b' > 0 \qquad b'^2 < 4a'c'$$

$$a' \propto J \quad b' \propto D \quad c' \propto K \qquad \frac{D^2}{JK} < \lambda_2$$

Heliknoton: Hopfion in a spiral background



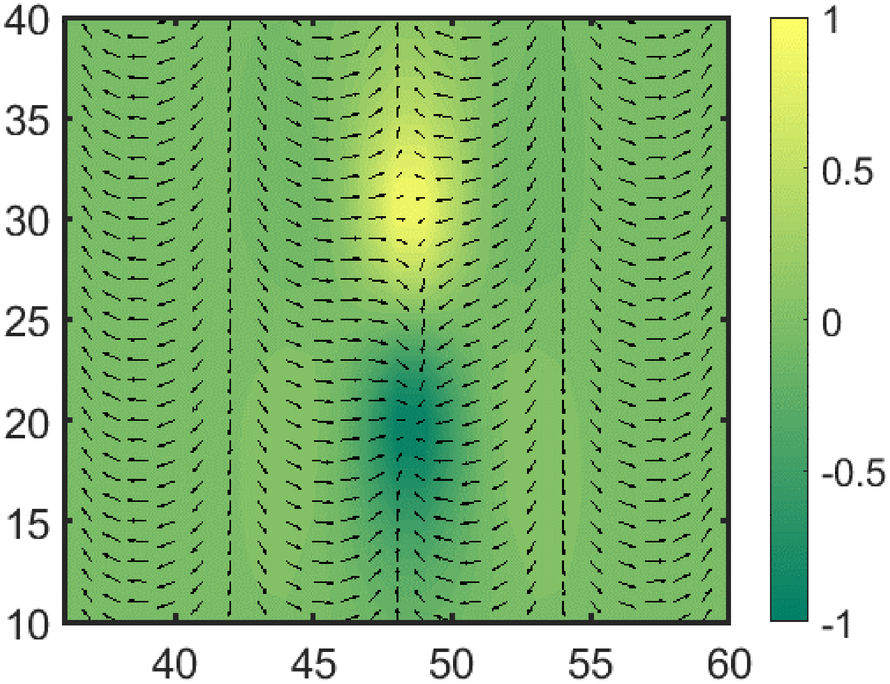
R. Voinescu et al. PRL 125, 057201 (2020)

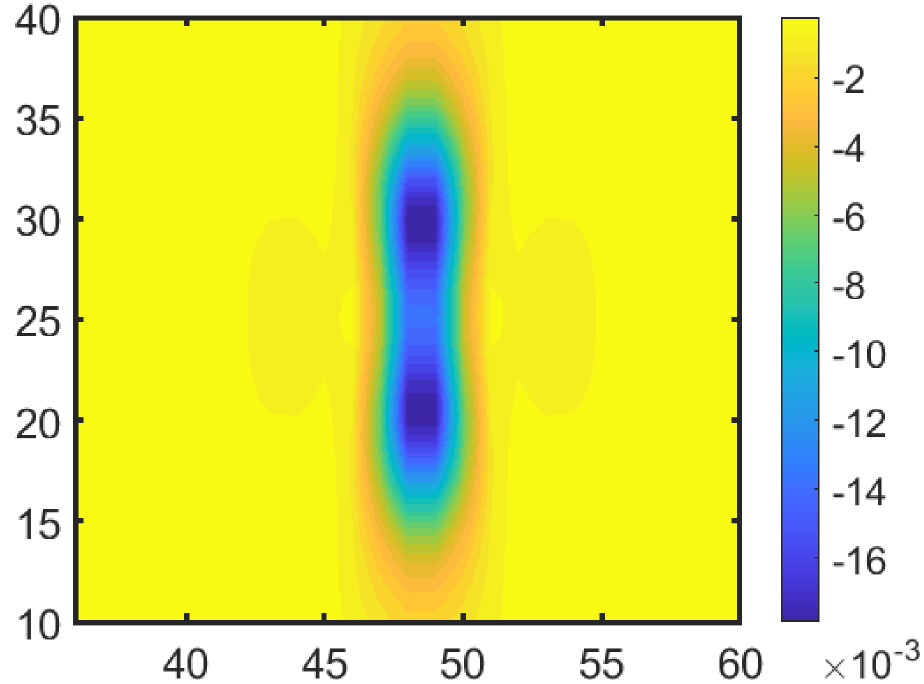


Long Li et al. Nature Materials 25, 950 (2026)

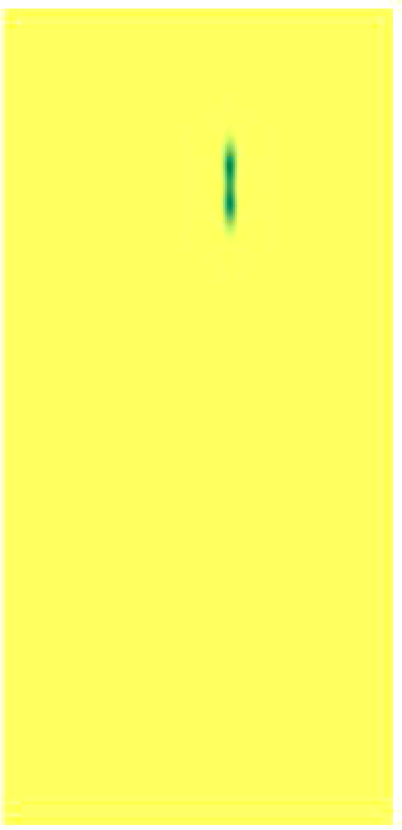
Bi-meron in spin-spiral

L. Maranzana, MM, Naoto Nagaosa and Sergey Artyukhin, arXiv:2502.13083





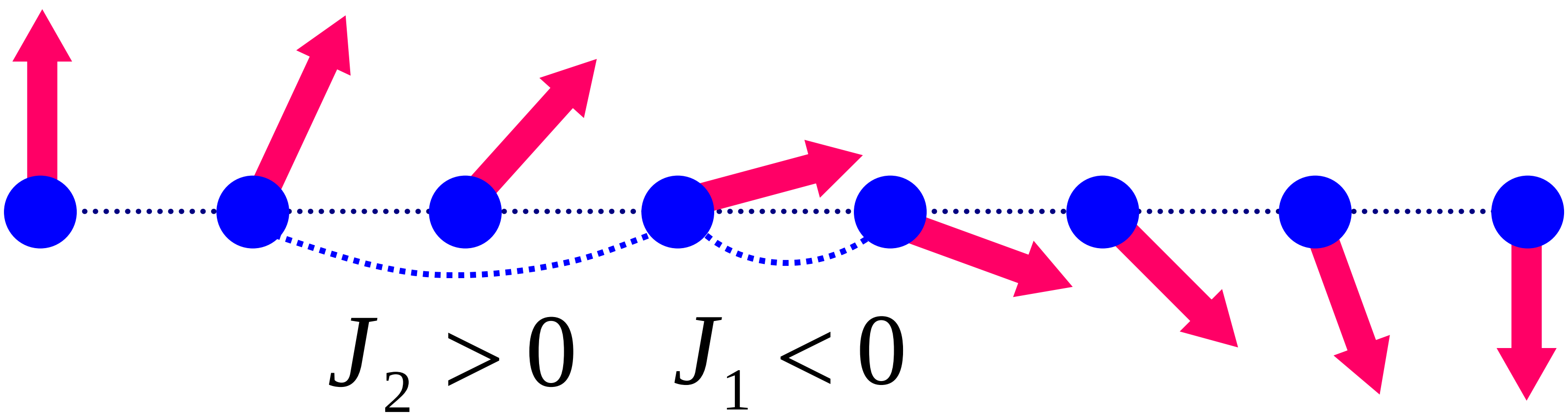
Bi-meron dynamics in rotating magnetic field



Skyrmions in frustrated magnets

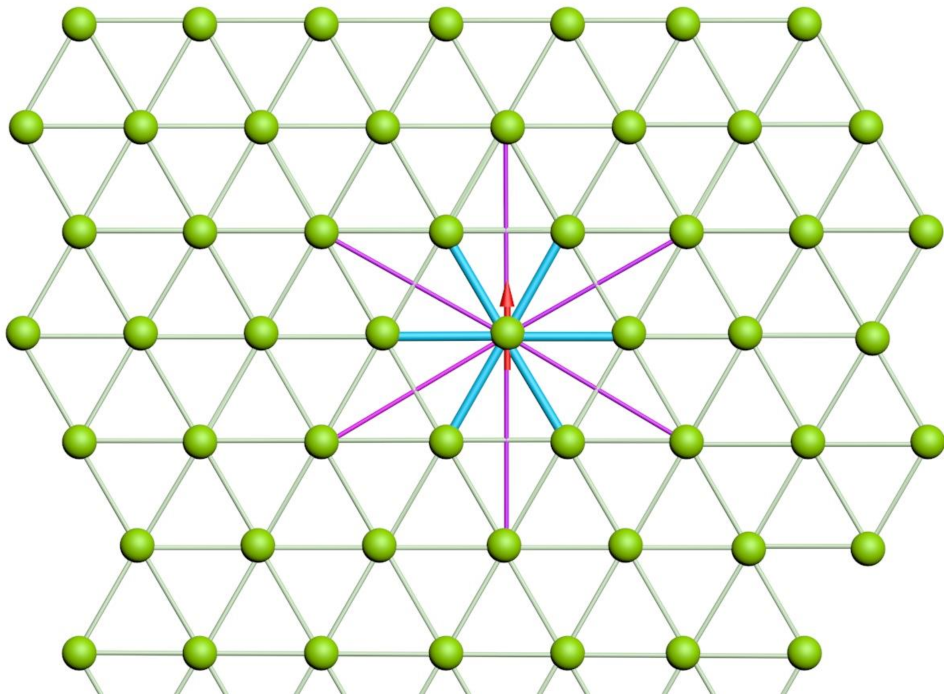
Spirals in centrosymmetric frustrated magnets

J. J. Villain, Phys. Chem. Solids 11, 303 (1959)
A. J. Yoshimori, Phys. Soc. Jpn. 14, 807 (1959)



Frustrated triangular magnet with anisotropy

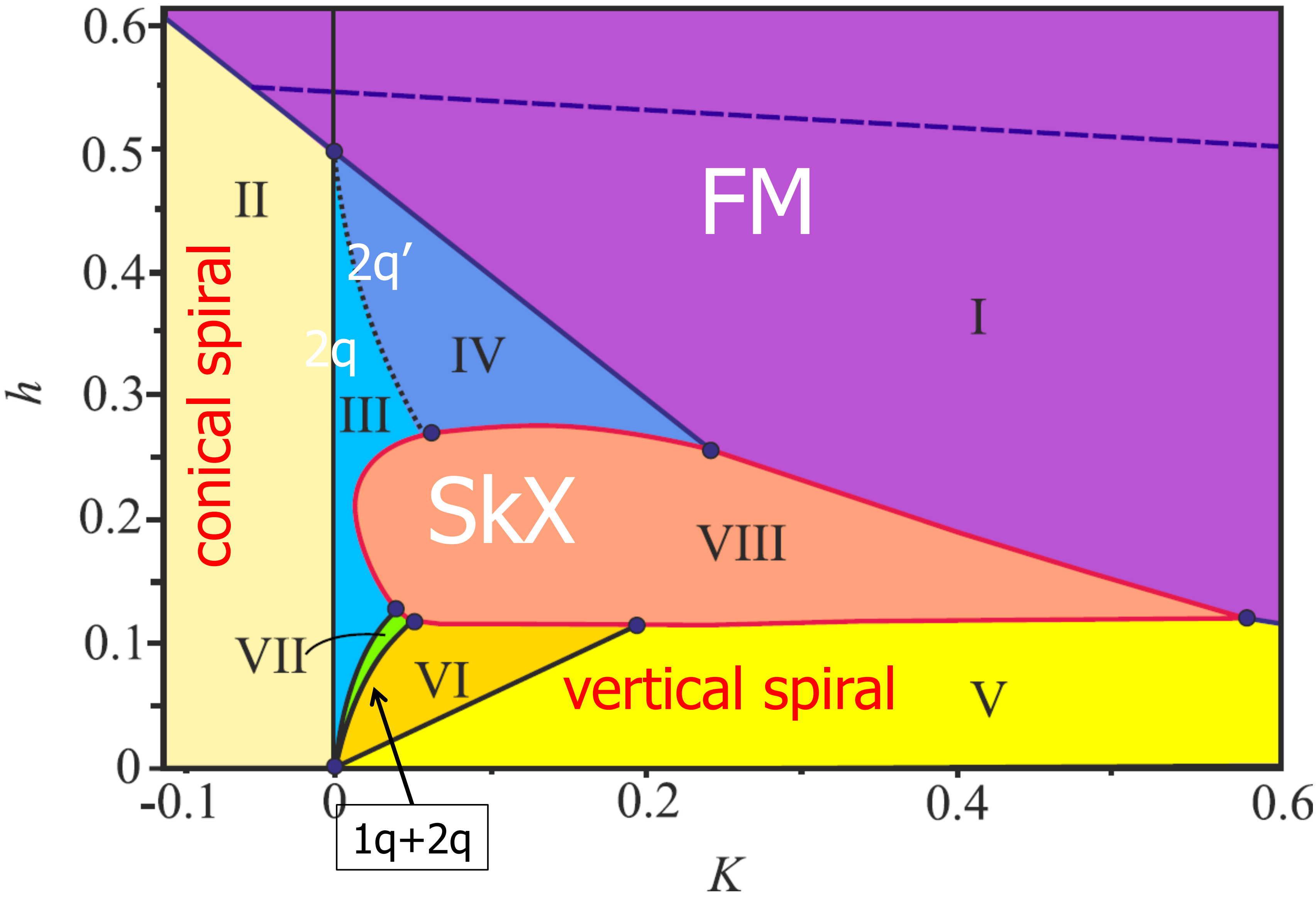
$$E = -J_1 \sum_{<i,j>} \boldsymbol{S}_i \boldsymbol{S}_j + J_2 \sum_{\ll i,j \gg} \boldsymbol{S}_i \boldsymbol{S}_j - h \sum_i S_{iz} - \frac{K}{2} \sum_i (S_{iz})^2$$



$Fe_xNi_{1-x}Br_2$

Phase Diagram

A. Leonov and MM, Nature Commun. ***6****, 8275 (2015)*



Hopfion stability in frustrated magnets

4th-order terms $$\int d^3x \partial_i \partial_j \boldsymbol{n} \cdot \partial_i \partial_j \boldsymbol{n}$$

Scaling $$E(R) = dR^{-1} + aR$$

$$R = \sqrt{\frac{d}{a}}$$

usually, $d \sim a l_0^2$

lattice constant

$$R \sim l_0$$

a must be very small → fine tuning

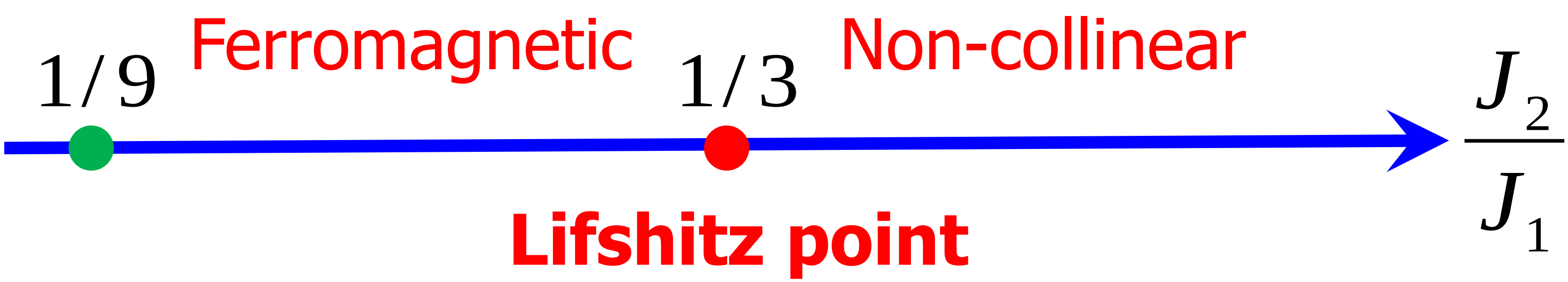
Continuum model of a frustrated magnet on a square lattice

$$E = \frac{1}{2}\int d^2x \left(\rho \partial_i \mathbf{n} \cdot \partial_i \mathbf{n} + \mu \Delta \mathbf{n} \cdot \Delta \mathbf{n} - 2h n_z - K n_z^2 \right)$$

Spin stiffness

$$\rho = \frac{3}{2}\left(J_1 - 3J_2\right) \qquad \mu = \frac{3}{32}\left(9J_2 - J_1\right)$$

$$\rho > 0 \qquad \rho < 0$$



Summary

- Topological magnetic defects in 1, 2 and 3 dimensions. Matching dimensionalities of order parameter and space.
- Emergent electromagnetism
- Nontrivial topology does not guarantee stability against collapse of topological defects
- Magnetic particles in (anti)ferromagnets