



Fields, magnetostatics, units.



Marco Coïsson
INRIM, Torino, Italy

WHAT ARE WE TALKING ABOUT TODAY?

- ▶ **Maxwell equations:** it might seem odd, but even though they are at the foundations of magnetism, you seldom use them directly.
- ▶ **Magnetic moments, dipoles:** in a more phenomenological approach, magnetic moments are useful tools to understand the fundamentals of magnetism.
- ▶ **Units:** because they are less obvious than it could seem (different similar but not identical quantities, different systems of units, ...).
- ▶ **Fields:** depending on how you look at things, you have applied fields, internal fields, demagnetising fields, magnetisations, magnetic inductions, etc. All are interdependent, not all have the same sources, not all can be measured, those that can require different techniques, ...

MAXWELL EQUATIONS

EVERYTHING STARTS HERE

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \cdot \vec{B} = 0$$

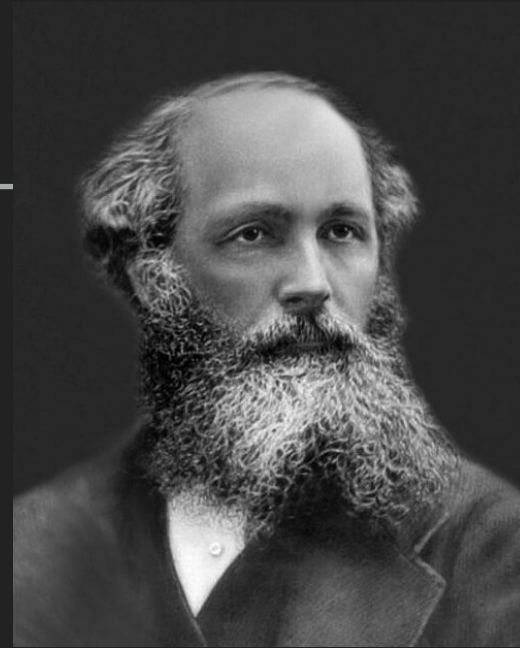
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

$$\vec{D} = \epsilon_0 \epsilon_r \vec{E}$$

$$\vec{B} = \mu_0 \mu_r \vec{H}$$

$$\mu_r = \mu_r(H)$$



EVERYTHING STARTS HERE

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{D} = \epsilon_0 \epsilon_r \vec{E}$$

$$\nabla \cdot \vec{B} = 0$$

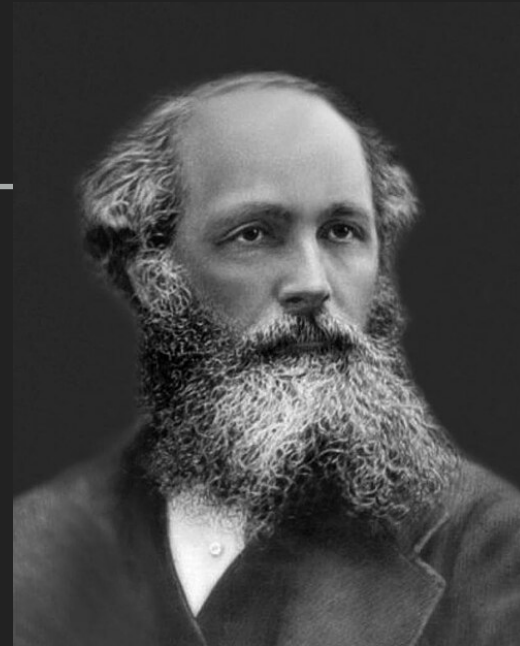
$$\vec{B} = \mu_0 \mu_r \vec{H}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\mu_r = \mu_r(H)$$

$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

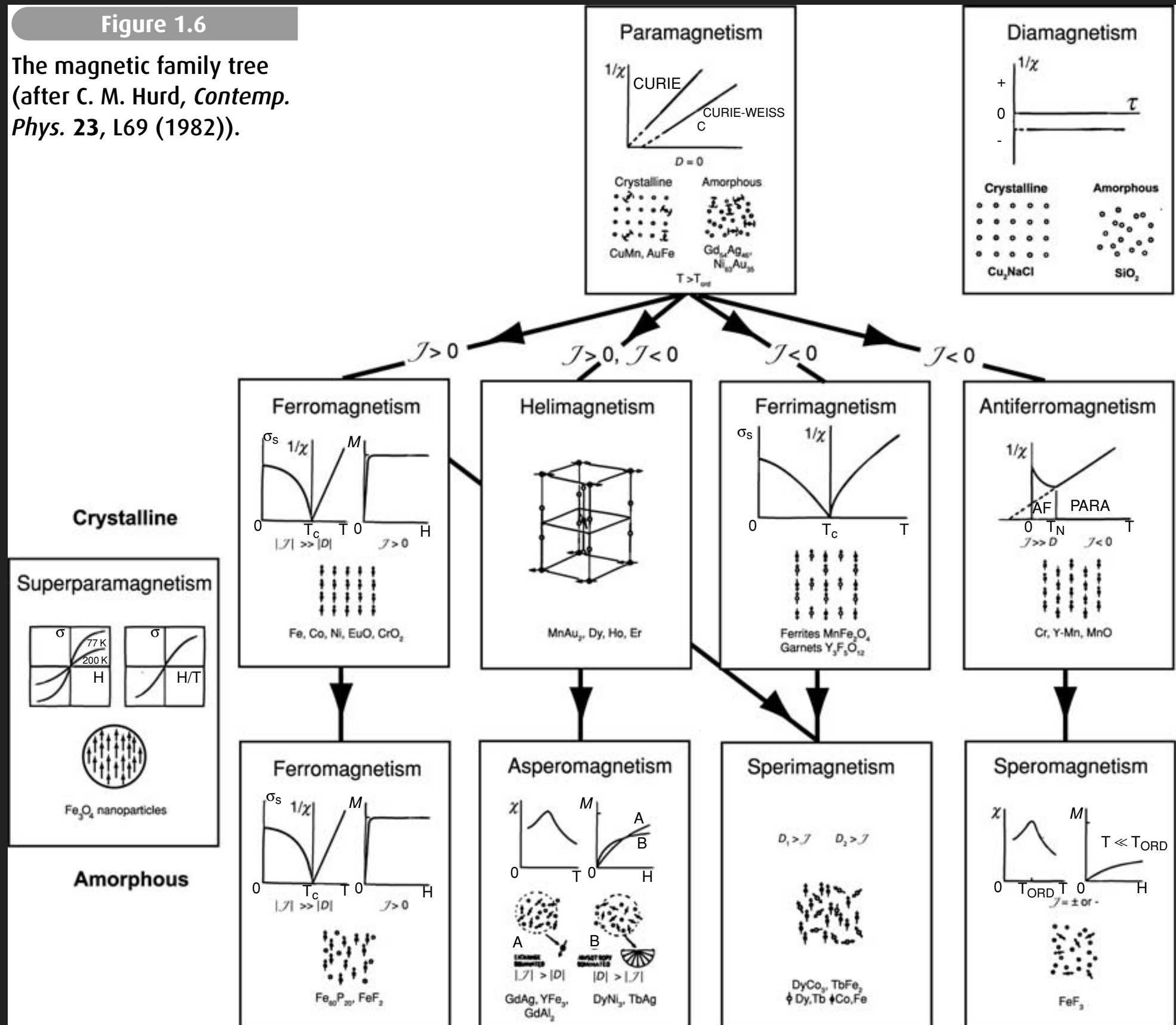
Now it's just a matter of describing what's this magnetic-field dependence of the relative permeability.



GOOD LUCK (THAT'S WHY WE NEED SCHOOLS OF MAGNETISM)

Figure 1.6

The magnetic family tree
(after C. M. Hurd, *Contemp. Phys.* 23, L69 (1982)).



PHENOMENOLOGICAL APPROACH

- ▶ Magnetic dipoles $\mathbf{m}$, and then $\mathbf{M}(\mathbf{r})$ is its mesoscopic volume average.
- ▶ $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$.
- ▶ Field produced by a given distribution of magnetisation: integral of the dipole field due to each volume element $\mathbf{M}(\mathbf{r})dV$.
- ▶ Scalar φ_m and vector $\mathbf{A}$ potentials are defined for $\mathbf{H}$ and $\mathbf{B}$ respectively.
- ▶ Internal, external and demagnetising fields are calculated.
- ▶ Magnetic forces and energies are derived from the fields.

MAGNETIC MOMENTS

MAGNETIC MOMENTS

- ▶ The **elementary quantity** in solid-state magnetism.
- ▶ Magnetic moments come from atoms (**electron spins, electron motion**, nuclear spin).
 - ▶ Use Quantum Mechanics to **add magnetic moments**.
 - ▶ You'll find out that almost all elements of the periodic table do not possess a resultant moment on the atomic scale in solids.

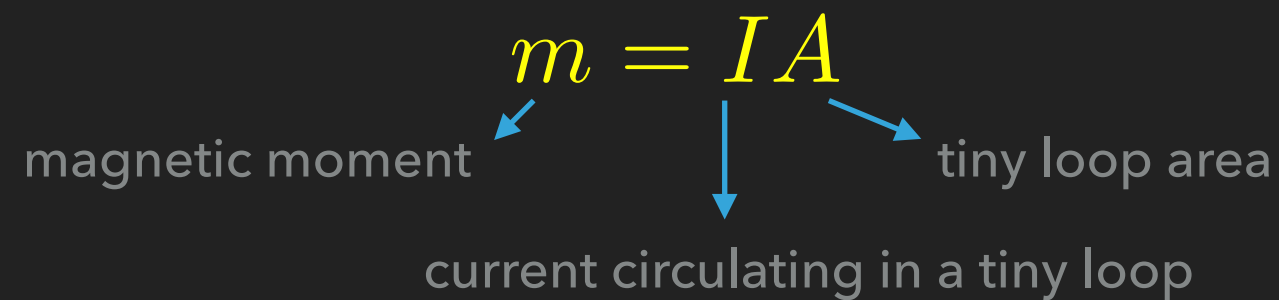
SO WHAT ARE THESE FANCY MAGNETIC MOMENTS?

- ▶ Quantum origin.
- ▶ A classical equivalent (Ampère equivalence): current loops.

$$m = IA$$

magnetic moment tiny loop area

current circulating in a tiny loop



$$[m] = Am^2$$

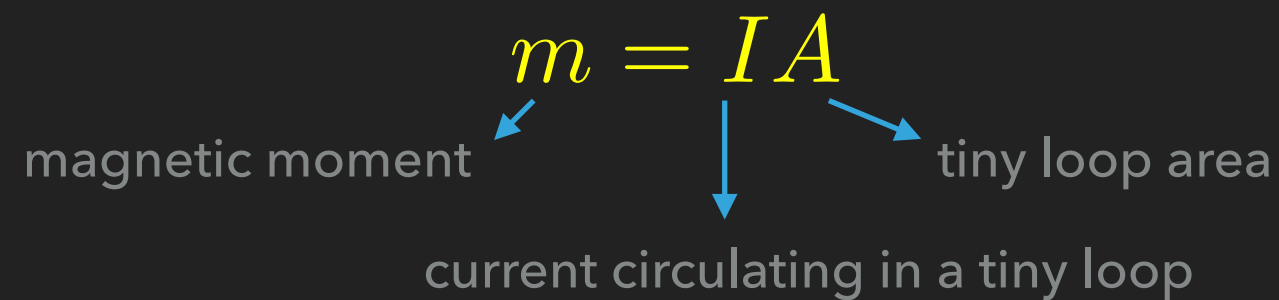
$$[\vec{M}] = \left[\frac{m}{V} \right] = \frac{A}{m}$$

SO WHAT ARE THESE FANCY MAGNETIC MOMENTS?

- ▶ Quantum origin.
- ▶ A classical equivalent (Ampère equivalence): **current loops.**

$$m = IA$$

magnetic moment current circulating in a tiny loop tiny loop area



it's only an equivalent: they do not exist!

$$[m] = Am^2$$

$$[\vec{M}] = \left[\frac{m}{V} \right] = \frac{A}{m}$$

UNITS

UNITS, OH, UNITS!

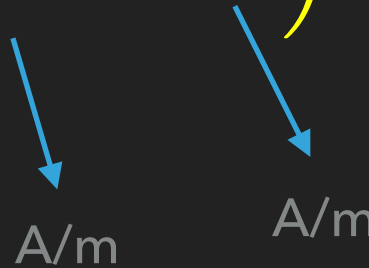
$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right)$$

UNITS, OH, UNITS!

$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right)$$

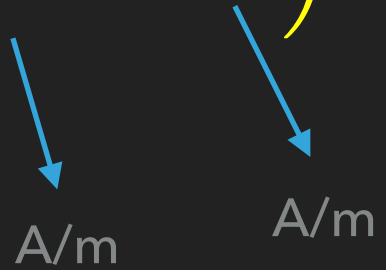

A/m

UNITS, OH, UNITS!

$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right)$$


The diagram shows two blue arrows pointing downwards from the vectors $\vec{H}$ and $\vec{M}$ in the equation above. Each arrow points to the unit A/m written in grey text below it.

UNITS, OH, UNITS!

$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right)$$


A/m A/m

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

UNITS, OH, UNITS!

$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right)$$

A/m A/m

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

1/m V/m

UNITS, OH, UNITS!

$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right)$$

A/m A/m

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

1/m V/m

$\frac{Vs}{m^2} = \frac{Wb}{m^2} = T$

UNITS, OH, UNITS!

$$\vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right)$$

A/m A/m

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}$$

1/m V/m

$\frac{Vs}{m^2} = \frac{Wb}{m^2} = T$

so H is in A/m, but $\mu_0 H$ is in T

UNITS, OH, UNITS!

The diagram illustrates the derivation of units for the permeability of free space, μ_0 , using Maxwell's equations. Key terms in the equations are circled in blue to highlight the units being tracked.

Equation 1: $\vec{B} = \mu_0 (\vec{H} + \vec{M})$

Units for $\vec{H}$ and $\vec{M}$ are both A/m .

Equation 2: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Units for ∇ is $1/m$ and for $\vec{E}$ is V/m .

Equation 3: $\frac{Vs}{m^2} = \frac{Wb}{m^2} = T$

Unit for T is derived from the previous equation.

Equation 4: $[\mu_0] = \frac{Tm}{A} = \frac{H}{m}$

Unit for μ_0 is derived from the previous equation.

Text: so H is in A/m , but $\mu_0 H$ is in T

HERE COMES THE HEADACHE...

- ▶ μ_0 is a **quantity**:
 - ▶ $1.256\,637\,061\,27(20) \times 10^{-6} \text{ H/m}$ (CODATA).
- ▶ μ_0 **used to be** a constant:
 - ▶ $4\pi \times 10^{-7} \text{ H/m} = 1.256\,637\,061\,44 \times 10^{-6} \text{ H/m}$.

HERE COMES THE HEADACHE...

▶ μ_0 is a **quantity**:  the result of a measurement
(since the revision of SI in 2019)

▶ $1.256\,637\,061\,27(20) \times 10^{-6} \text{ H/m}$ (CODATA).

▶ μ_0 **used to be** a constant:

▶ $4\pi \times 10^{-7} \text{ H/m} = 1.256\,637\,061\,44 \times 10^{-6} \text{ H/m}$.

HERE COMES THE HEADACHE...

▶ μ_0 is a **quantity**: → the result of a measurement
(since the revision of SI in 2019)

▶ $1.256\,637\,061\,27(20) \times 10^{-6} \text{ H/m}$ (CODATA).

▶ μ_0 **used to be** a **constant**: → the result of a definition

▶ $4\pi \times 10^{-7} \text{ H/m} = 1.256\,637\,061\,44 \times 10^{-6} \text{ H/m}$.

HERE COMES THE HEADACHE...

▶ μ_0 is a **quantity**: → the result of a measurement
(since the revision of SI in 2019)

▶ $1.256\,637\,061\,27(20) \times 10^{-6} \text{ H/m}$ (CODATA).

▶ μ_0 **used to be** a **constant**: → the result of a definition

▶ $4\pi \times 10^{-7} \text{ H/m} = 1.256\,637\,061\,44 \times 10^{-6} \text{ H/m}$.

For any practical purposes, the conceptual difference is philosophical, and the numerical difference is lower than any experimental uncertainty. But...

"Nothing has changed, but nothing's the same"

David Byrne

THE DEATH OF (THE) GAUSS

- ▶ μ_0 is a **quantity**:
 - ▶ as a result of an experiment, μ_0 does **not** have a fixed value.
 - ▶ it is no longer possible to give it an arbitrary value to make your life easier.
- ▶ In the (old) CGS system, $\mu_0 = 1$ by definition.
 - ▶ Conceptually, units of magnetic induction (G, gauss) expressed in the CGS system are no longer valid!
- ▶ However...

CGS UNITS

- ▶ CGS: cm, g, s.
- ▶ CGS equivalent of current = 10 A.
- ▶ CGS equivalent of voltage = 10 nV.
- ▶ Energy in **erg** = 10^{-7} J.
- ▶ Magnetic field **H** in Oe (oersted).
- ▶ Magnetic induction **B** in G (gauss) = emu / cm³.
- ▶ **$B = H + 4\pi M$** , magnetisation **M** in emu/cm³.
- ▶ So μ_0 is a constant = 1 **but it has units!**
- ▶ 1 G = 1 Oe numerically, but **different units!**

[\[https://commons.wikimedia.org/wiki/File:%C3%98rsted.jpg\]](https://commons.wikimedia.org/wiki/File:%C3%98rsted.jpg)

[\[https://commons.wikimedia.org/wiki/File:Carl_Friedrich_Gauss.jpg\]](https://commons.wikimedia.org/wiki/File:Carl_Friedrich_Gauss.jpg)

[\[https://commons.wikimedia.org/wiki/File:Dromaius_novaehollandiae_-_R%C3%A9serve_africaine_de_Sigean_01.jpg\]](https://commons.wikimedia.org/wiki/File:Dromaius_novaehollandiae_-_R%C3%A9serve_africaine_de_Sigean_01.jpg)

CGS UNITS

- ▶ CGS: cm, g, s.
- ▶ CGS equivalent of current = 10 A.
- ▶ CGS equivalent of voltage = 10 nV.
- ▶ Energy in **erg** = 10^{-7} J.
- ▶ Magnetic field **H** in Oe (oersted).
- ▶ Magnetic induction **B** in G (gauss) = emu / cm³.
- ▶ **$B = H + 4\pi M$** , magnetisation **M** in emu/cm³.
- ▶ So **μ_0** is a constant = 1 **but it has units!**
- ▶ 1 G = 1 Oe numerically, but **different units!**



[<https://commons.wikimedia.org/wiki/File:%C3%98rsted.jpg>]

[https://commons.wikimedia.org/wiki/File:Carl_Friedrich_Gauss.jpg]

[https://commons.wikimedia.org/wiki/File:Dromaius_novaehollandiae_-_R%C3%A9serve_africaine_de_Sigean_01.jpg]

CGS UNITS

- ▶ CGS: cm, g, s.
- ▶ CGS equivalent of current = 10 A.
- ▶ CGS equivalent of voltage = 10 nV.
- ▶ Energy in **erg** = 10^{-7} J.
- ▶ Magnetic field **H** in Oe (oersted).
- ▶ Magnetic induction **B** in G (gauss) = emu / cm³.
- ▶ **$B = H + 4\pi M$** , magnetisation **M** in emu/cm³.
- ▶ So μ_0 is a constant = 1 **but it has units!**
- ▶ 1 G = 1 Oe numerically, but **different units!**



[<https://commons.wikimedia.org/wiki/File:%C3%98rsted.jpg>]

[https://commons.wikimedia.org/wiki/File:Carl_Friedrich_Gauss.jpg]

[https://commons.wikimedia.org/wiki/File:Dromaius_novaehollandiae_-_R%C3%A9serve_africaine_de_Sigean_01.jpg]

CGS UNITS

- ▶ CGS: cm, g, s.
- ▶ CGS equivalent of current = 10 A.
- ▶ CGS equivalent of voltage = 10 nV.
- ▶ Energy in **erg** = 10^{-7} J.
- ▶ Magnetic field **H** in Oe (oersted).
- ▶ Magnetic induction **B** in G (gauss) = emu / cm³.
- ▶ **$B = H + 4\pi M$** , magnetisation **M** in emu/cm³.
- ▶ So **μ_0** is a constant = 1 **but it has units!**
- ▶ 1 G = 1 Oe numerically, but **different units!**



[<https://commons.wikimedia.org/wiki/File:%C3%98rsted.jpg>]

[https://commons.wikimedia.org/wiki/File:Carl_Friedrich_Gauss.jpg]

[https://commons.wikimedia.org/wiki/File:Dromaius_novaehollandiae_-_R%C3%A9serve_africaine_de_Sigean_01.jpg]

CONVERTING UNITS

Quantity	Symbol	CGS	Conversion	SI
Magnetic flux density, magnetic induction	B	gauss (G)	10^{-4}	tesla T, Wb/m ²
Magnetic flux	Φ	maxwell (Mx), G cm ²	10^{-8}	weber Wb, Vs
Magnetic potential difference, magnetomotive force	U, F	gilbert (Gb)	$10/4\pi$	ampere A
Magnetic field strength	H	oersted Oe, Gb/cm	$10^3/4\pi$	A/m
(Volume) magnetisation	M	emu/cm ³	10^3	A/m
(Volume) magnetisation	$4\pi M$	G	$10^3/4\pi$	A/m
Magnetic polarisation	J, I	emu/cm ³	$4\pi \times 10^{-4}$	T, Wb/m ²
(Mass) magnetisation	σ , M	emu/g	1	Am ² / kg
Magnetic moment	m	emu, erg/G	10^{-3}	Am ² , J/T
Magnetic dipole moment	j	emu, erg/G	$4\pi \times 10^{-10}$	Wb m
(Volume) susceptibility	χ , κ	no dim., emu/cm ³	4π $(4\pi)^2 \times 10^{-7}$	no dim H/m, Wb/(Am)
(Mass) susceptibility	χ_p , κ_p	cm ³ /g emu/g	$4\pi \times 10^{-3}$ $(4\pi)^2 \times 10^{-10}$	m ³ / kg Hm ² / kg

CONVERTING UNITS (CONT.)

Quantity	Symbol	CGS	Conversion	SI
(Molar) susceptibility	$\chi_{\text{mol}}, \kappa_{\text{mol}}$	$\text{cm}^3/\text{mol}, \text{emu}/\text{mol}$	$4\pi \times 10^{-6}$ $(4\pi)^2 \times 10^{-13}$	m^3 / mol Hm^2 / mol
Permeability	μ	no dim.	$4\pi \times 10^{-7}$	$\text{H}/\text{m}, \text{Wb}/(\text{Am})$
Relative permeability	μ_r	not defined		no dim.
(Volume) energy density	W	erg/cm^3	10^{-1}	J/m^3
Demagnetisation factor	D, N	no dim.	$1/4\pi$	no dim.

multiply number in _____ by _____ to convert it to _____

MAGNETISATION

MESOSCOPIC AVERAGES

- ▶ Local magnetic moment density: $\mathbf{M}(\mathbf{r}, t)$.
- ▶ Fluctuations on $< \text{nm}$ scale: the minimum length scale of the fluctuations in a ferromagnet is the exchange length ($\sim$ a few nm).
- ▶ Fluctuations on $< \text{ns}$ scale.
- ▶ Steady, homogeneous, local magnetisation averaging over a few nm and a few μs :

$$\vec{M}(\vec{r}) = \left\langle \vec{M}(\vec{r}, t) \right\rangle$$

Continuous medium approximation.

MACROSCOPIC AVERAGES

$$\vec{M} = \sum_i \frac{\vec{M}_i V_i}{\sum_i V_i}$$

sample volume

MACROSCOPIC AVERAGES

$$\vec{M} = \sum_i \frac{\vec{M}_i V_i}{\sum_i V_i}$$

sample volume

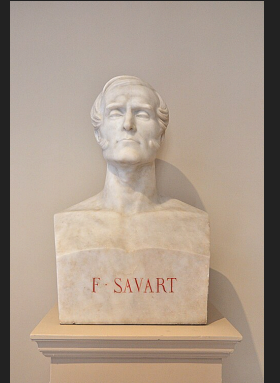
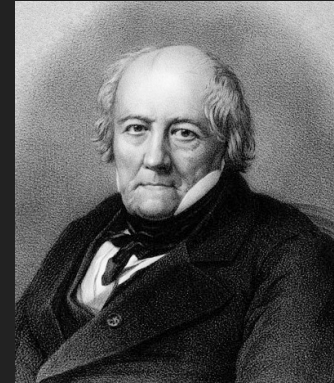
its value as a function of the applied field
is the hysteresis loop

FIELDS

IN THE BEGINNING...

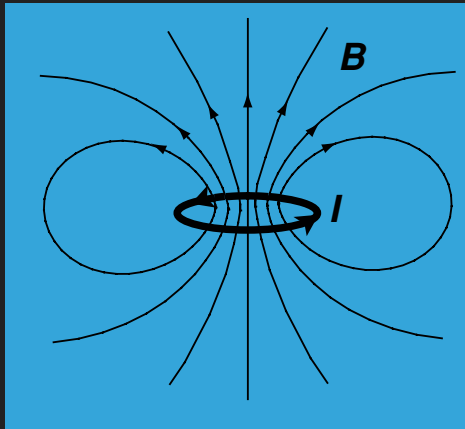
- ▶ Everything begins with the good old Biot-Savart law:

$$\delta \vec{B} = -\frac{\mu_0}{4\pi} \frac{\vec{r} \times \vec{j}}{r^3} \delta V$$



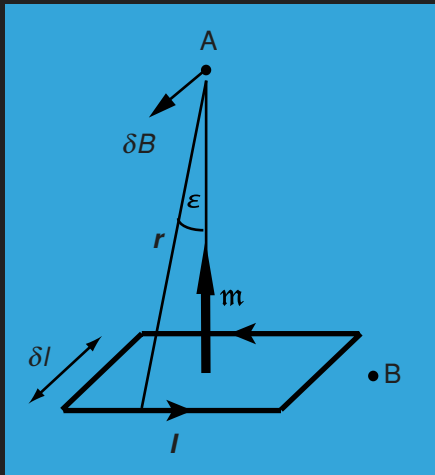
- ▶ which is the field generated by a small current element $\vec{j}\delta V$ at a point $\mathbf{P}$ of coordinates $\mathbf{r}$.
- ▶ Then you have arbitrary distributions of currents...
- ▶ ... flowing in arbitrary closed circuits...
- ▶ ... and you enjoy spending your time solving integrals...

FIELD OF A LOOP



$$B_0 = \frac{\mu_0 I}{2a}$$

at the centre of a circular loop.



$$B_A = 2 \frac{\mu_0 m}{4\pi r^3}$$

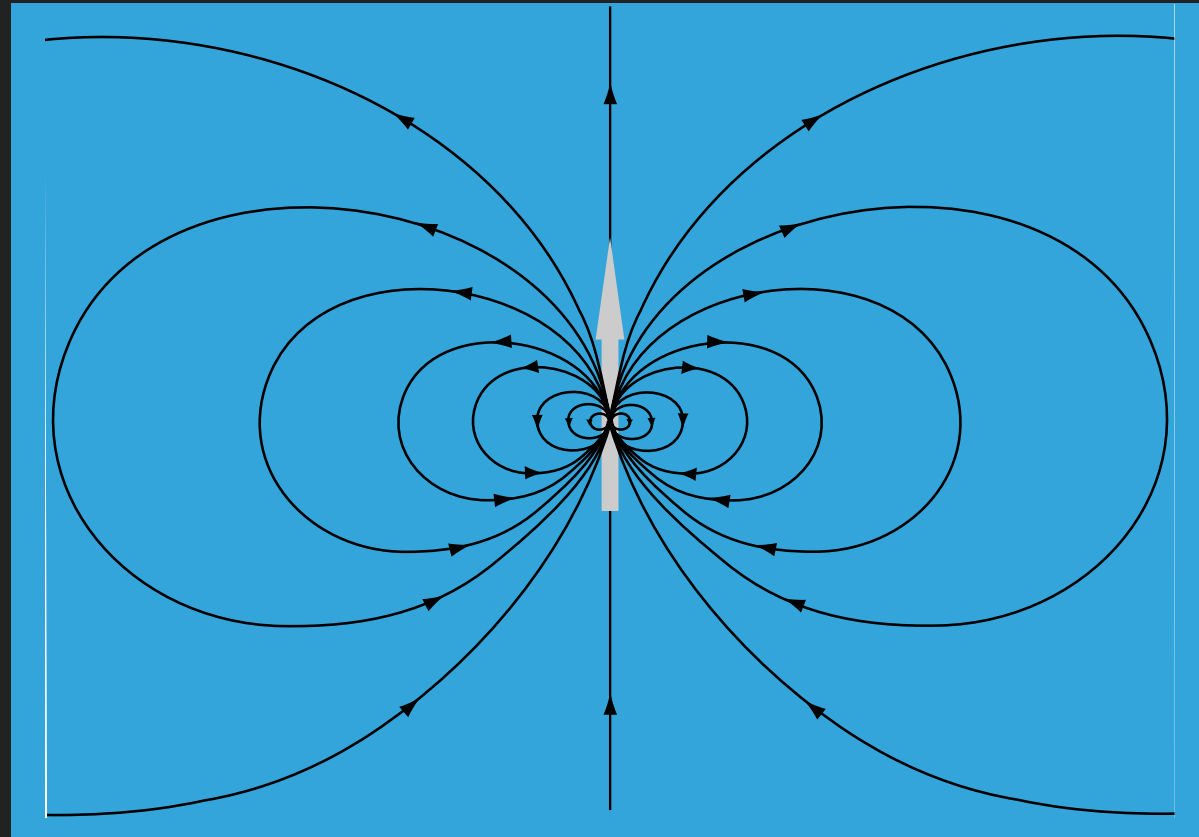
along the axis of a square loop.

$$B_B = \frac{\mu_0 m}{4\pi r^3}$$

half as large as in point A, but with opposite direction

MAGNETIC DIPOLE

And for the generic point anywhere far away from the magnetic moment?



Mathematically, the equation of the lines of force of a magnetic moment has the same form as that of an **electric dipole**.

By analogy, **m** is also called a **magnetic dipole**, and the field that it generates is the **magnetic dipole field**.

MAGNETIC DIPOLE FIELD

(The full calculations are left as an easy exercise for the reader)

$$\vec{B} = \frac{\mu_0 m}{4\pi r^5} \left[3xz\vec{e}_x + 3yz\vec{e}_y + (3z^2 - r^2)\vec{e}_z \right]$$

in Cartesian coordinates with $\mathbf{m} \parallel \mathbf{e}_z$

$$\vec{B} = \frac{\mu_0}{4\pi} \left[3 \frac{(\vec{m} \cdot \vec{r}) \vec{r}}{r^5} - \frac{\vec{m}}{r^3} \right]$$

in components // $\mathbf{r}$ and $\mathbf{m}$.

The dipolar field in the far field approximation $\sim 1/r^3$.

**MANY DIFFERENT
MAGNETIC FIELDS**

DIFFERENT FIELDS, WITH DIFFERENT UNITS AND DIMENSIONS

- ▶ **B**: the induction. So far we have used it.
- ▶ **H**: the magnetic field, why would we need it anyway?
- ▶ **M**: the magnetisation of the material, normally you plot it as a function of **H**.

Do we really need them all?

B

$$\nabla \cdot \vec{B} = 0 \quad \text{solenoidal field, no divergence!}$$

No magnetic monopoles (rest in peace, P.A.M. Dirac!) $q_m \approx \frac{h}{\mu_0 e}$

The net flux Φ of B across any closed surface S is exactly 0 $\int_S \vec{B} dA = 0$

$$[\Phi] = Wb, [\vec{B}] = \frac{Wb}{m^2} = T \quad B = \text{magnetic flux density or magnetic induction}.$$

(BTW, magnetic flux is quantised in superconducting circuits, stay tuned!)

SOURCES OF **B**

Electric currents are sources of **B**, including loops (and therefore magnetic moments **m**)

$$\nabla \times \vec{B} = \mu_0 \vec{j} \Leftrightarrow \oint \vec{B} dl = \mu_0 I$$

All sources of **B** are moving charges, and **B** interacts with electrical charges only when they move (Lorentz force).

$$\vec{f} = q \left(\vec{E} + \vec{v} \times \vec{B} \right)$$

Magnetic units are not fundamental!
They can be expressed in terms of fundamental SI units kg, m, s, A.

SOURCES OF $\mathbf{B}$

Electric currents are sources of $\mathbf{B}$, including loops (and therefore magnetic moments $\mathbf{m}$)

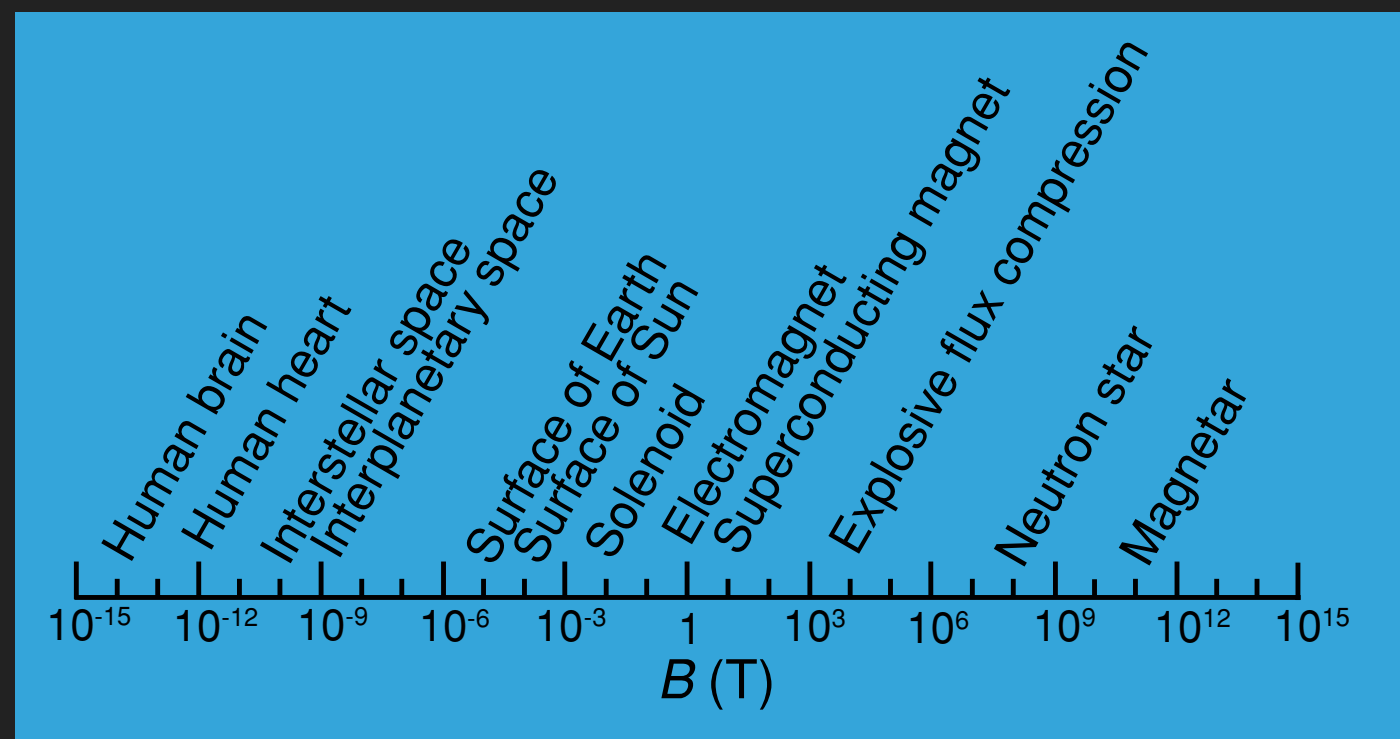
$$\nabla \times \vec{B} = \mu_0 \vec{j} \Leftrightarrow \oint \vec{B} dl = \mu_0 I$$

All sources of $\mathbf{B}$ are moving charges, and $\mathbf{B}$ interacts with electrical charges only when they move (Lorentz force).

$$\vec{f} = q \left(\vec{E} + \vec{v} \times \vec{B} \right)$$

Magnetic units are not fundamental!

They can be expressed in terms of fundamental SI units kg, m, s, A.



GENERATION OF B

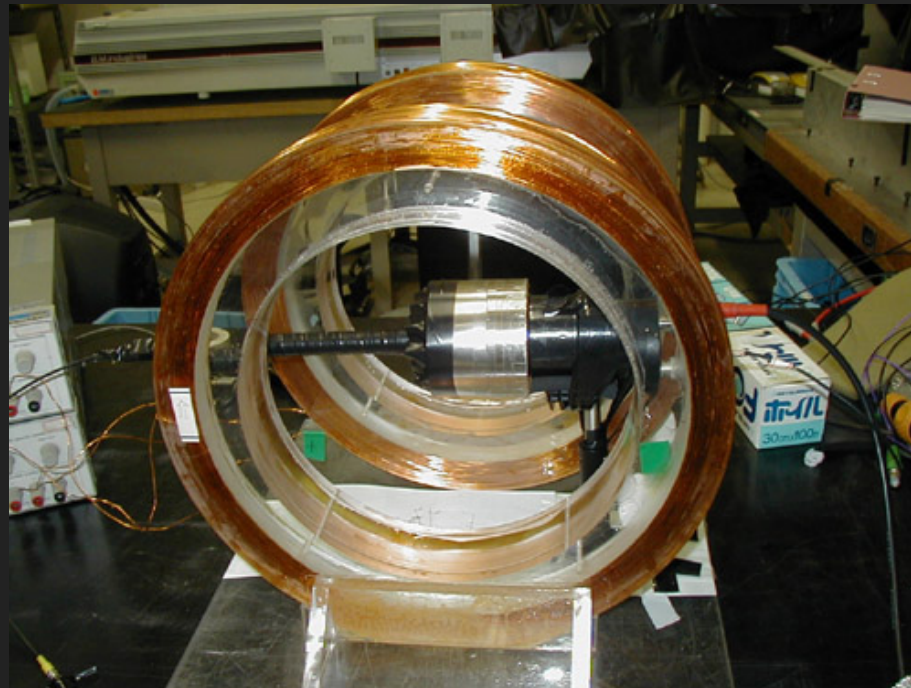
[<https://commons.wikimedia.org/wiki/File:AGEM5520.jpg>]
[https://commons.wikimedia.org/wiki/File:Helmholtz_coil,_Brookhaven_National_Laboratory.jpg]
[<https://cryomagnetics.com/products/high-field-nb3sn-solenoids/>]
[<https://www.hsmagnets.com/product/cylindrical-halbach-array-magnet/>]



Electromagnets

up to 2 – 3 T (20 – 30 kOe)

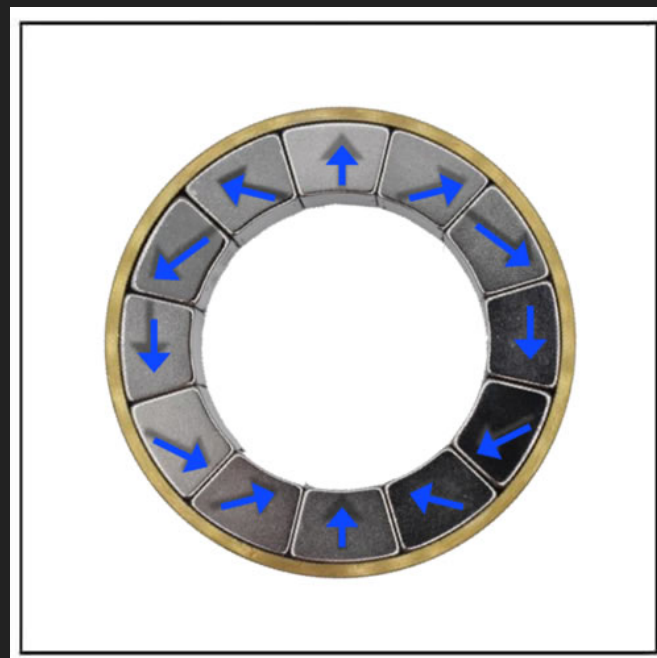
non-linear (closed-loop feedback required)



Helmholtz coils and solenoids
up to a few mT (~ 100 Oe)
linear (one-time calibration
required)



Superconducting solenoids
up to several T (< 20 T, < 200 kOe)
almost linear (one-time calibration
required) but with hysteresis



Halbach cylinders
up to ~ 2 T
no power supply required

H

- ▶ The **field strength** or **magnetising force**.
- ▶ The magnetisation of a material reflects the local value of H.
- ▶ In free space **$\mathbf{B} = \mu_0 \mathbf{H}$** .
- ▶ The problem arises when there are materials around.

H WHEN THERE ARE MATERIALS

$$\nabla \times \vec{B} = \mu_0 \vec{j} = \mu_0 (\vec{j}_c + \vec{j}_m)$$

Current in electrical conductors.
Can be measured.

Current associated to the magnetised medium.
Cannot be measured (mostly it is a
representation of the spins of the electrons of
the medium).

H WHEN THERE ARE MATERIALS

$$\nabla \times \vec{B} = \mu_0 \vec{j} = \mu_0 (\vec{j}_c + \vec{j}_m)$$

Current in electrical conductors.
Can be measured.

Current associated to the magnetised medium.
Cannot be measured (mostly it is a
representation of the spins of the electrons of
the medium).

$$\vec{j}_m = \nabla \times \vec{M} \quad (\text{unsurprisingly})$$

H WHEN THERE ARE MATERIALS

$$\nabla \times \vec{B} = \mu_0 \vec{j} = \mu_0 (\vec{j}_c + \vec{j}_m)$$

Current in electrical conductors.
Can be measured.

Current associated to the magnetised medium.
Cannot be measured (mostly it is a
representation of the spins of the electrons of
the medium).

$$\vec{j}_m = \nabla \times \vec{M} \quad (\text{unsurprisingly})$$

Let's define a new field $\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$

H WHEN THERE ARE MATERIALS

$$\nabla \times \vec{B} = \mu_0 \vec{j} = \mu_0 (\vec{j}_c + \vec{j}_m)$$

Current in electrical conductors.
Can be measured.

Current associated to the magnetised medium.
Cannot be measured (mostly it is a
representation of the spins of the electrons of
the medium).

$$\vec{j}_m = \nabla \times \vec{M} \quad (\text{unsurprisingly})$$

Let's define a new field $\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$

$$\nabla \times \vec{H} = \frac{1}{\mu_0} \nabla \times \vec{B} - \nabla \times \vec{M} = \vec{j}_c$$

H WHEN THERE ARE MATERIALS

$$\nabla \times \vec{H} = \vec{j}_c \quad \text{in differential form}$$

$$\oint \vec{H} \cdot d\vec{l} = I_c \quad \text{in integral form, along the path of the circuit where conduction currents flow}$$

H WHEN THERE ARE MATERIALS

$$\nabla \times \vec{H} = \vec{j}_c \quad \text{in differential form}$$

$$\oint \vec{H} \cdot d\vec{l} = I_c \quad \text{in integral form, along the path of the circuit where conduction currents flow}$$

$$\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$$

H WHEN THERE ARE MATERIALS

$$\nabla \times \vec{H} = \vec{j}_c \quad \text{in differential form}$$

$$\oint \vec{H} \cdot d\vec{l} = I_c \quad \text{in integral form, along the path of the circuit where conduction currents flow}$$

$$\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$$

$$\nabla \cdot \vec{H} = \frac{1}{\mu_0} \nabla \cdot \vec{B} - \nabla \cdot \vec{M}$$

H WHEN THERE ARE MATERIALS

$$\nabla \times \vec{H} = \vec{j}_c \quad \text{in differential form}$$

$$\oint \vec{H} \cdot d\vec{l} = I_c \quad \text{in integral form, along the path of the circuit where conduction currents flow}$$

$$\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$$

$$\nabla \cdot \vec{H} = \frac{1}{\mu_0} \nabla \cdot \vec{B} - \nabla \cdot \vec{M}$$

$$\nabla \cdot \vec{B} = 0$$

H WHEN THERE ARE MATERIALS

$$\nabla \times \vec{H} = \vec{j}_c \quad \text{in differential form}$$

$$\oint \vec{H} \cdot d\vec{l} = I_c \quad \text{in integral form, along the path of the circuit where conduction currents flow}$$

$$\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$$

$$\nabla \cdot \vec{H} = \frac{1}{\mu_0} \nabla \cdot \vec{B} - \nabla \cdot \vec{M}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{H} = -\nabla \cdot \vec{M}$$

H WHEN THERE ARE MATERIALS

$$\nabla \times \vec{H} = \vec{j}_c \quad \text{in differential form}$$

$$\oint \vec{H} \cdot d\vec{l} = I_c \quad \text{in integral form, along the path of the circuit where conduction currents flow}$$

$$\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$$

$$\nabla \cdot \vec{H} = \frac{1}{\mu_0} \nabla \cdot \vec{B} - \nabla \cdot \vec{M}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \cdot \vec{H} = -\nabla \cdot \vec{M}$$

Mmmmmmm... tasty. What does it mean?

THAT'S WHY A MAGNET WORKS, AFTER ALL

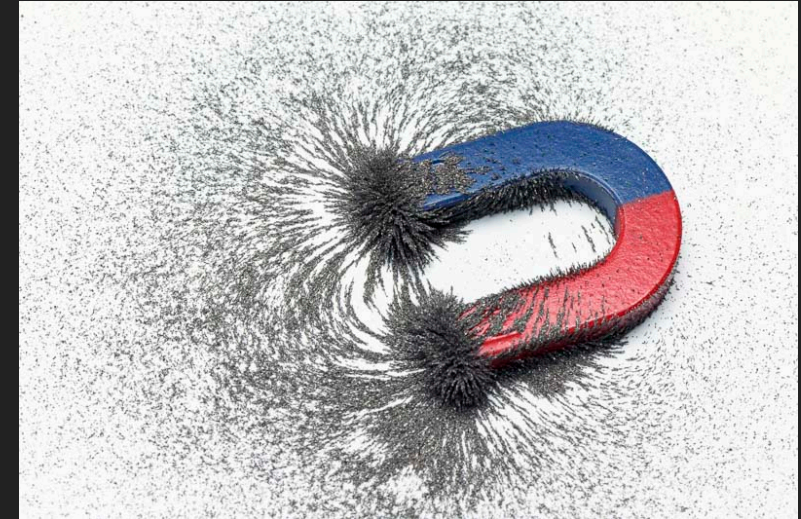
$$\nabla \cdot \vec{H} = -\nabla \cdot \vec{M}$$

H has a divergence that's not zero! It has sources and sinks associated to the non-uniformity of the magnetisation.

THAT'S WHY A MAGNET WORKS, AFTER ALL

$$\nabla \cdot \vec{H} = -\nabla \cdot \vec{M}$$

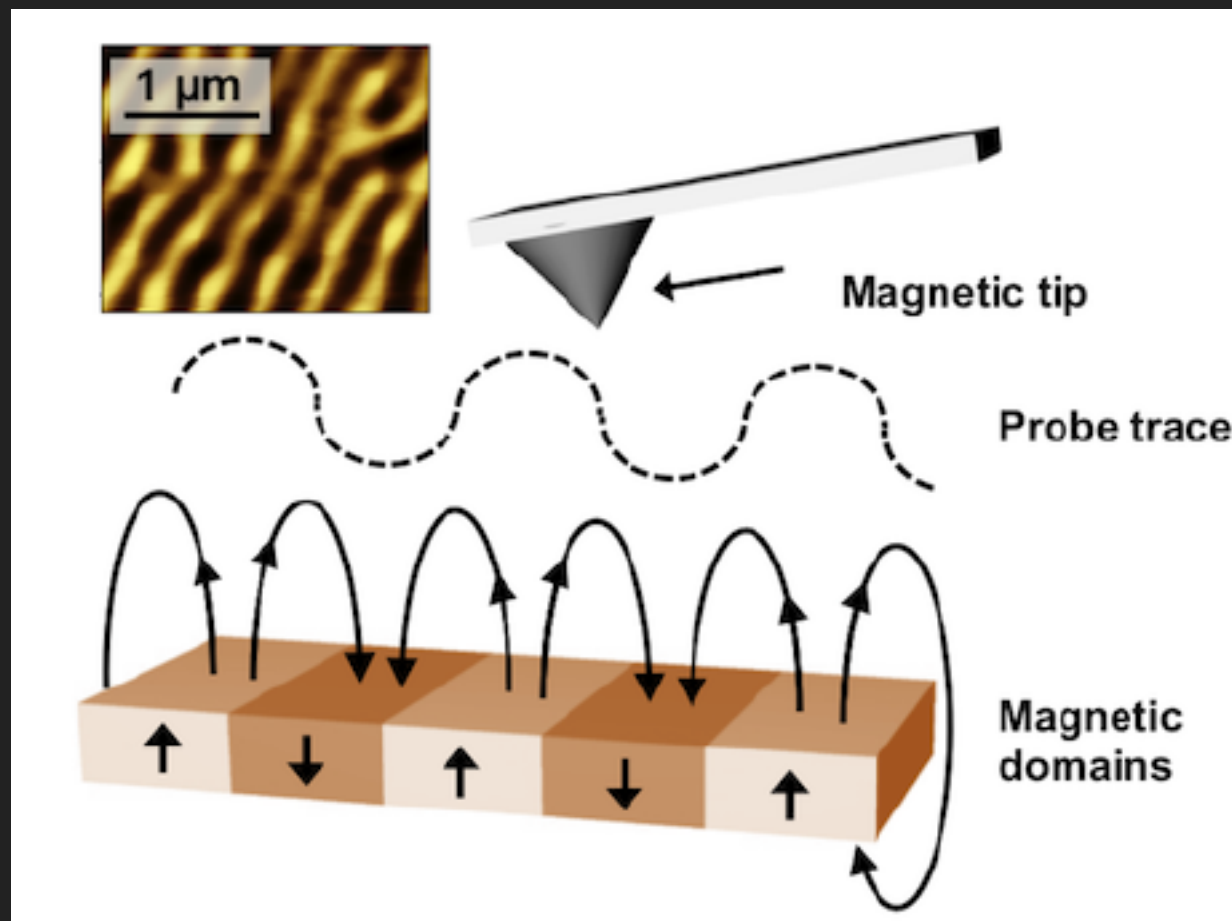
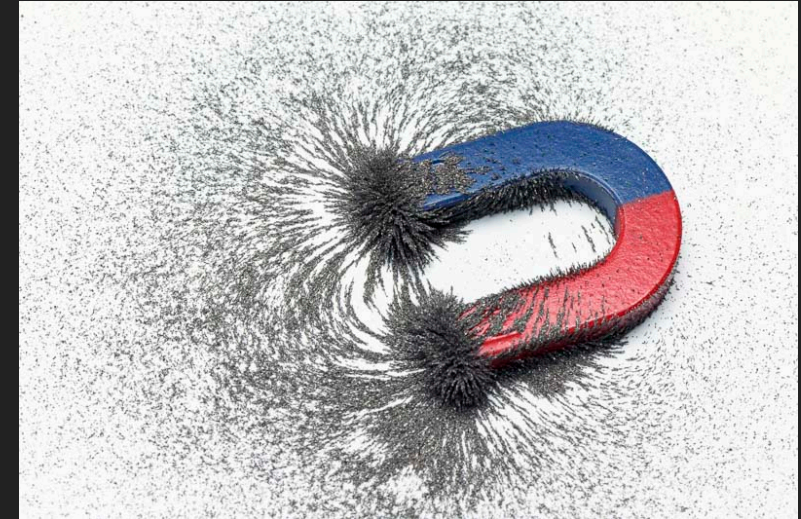
H has a divergence that's not zero! It has sources and sinks associated to the non-uniformity of the magnetisation.



THAT'S WHY A MAGNET WORKS, AFTER ALL

$$\nabla \cdot \vec{H} = -\nabla \cdot \vec{M}$$

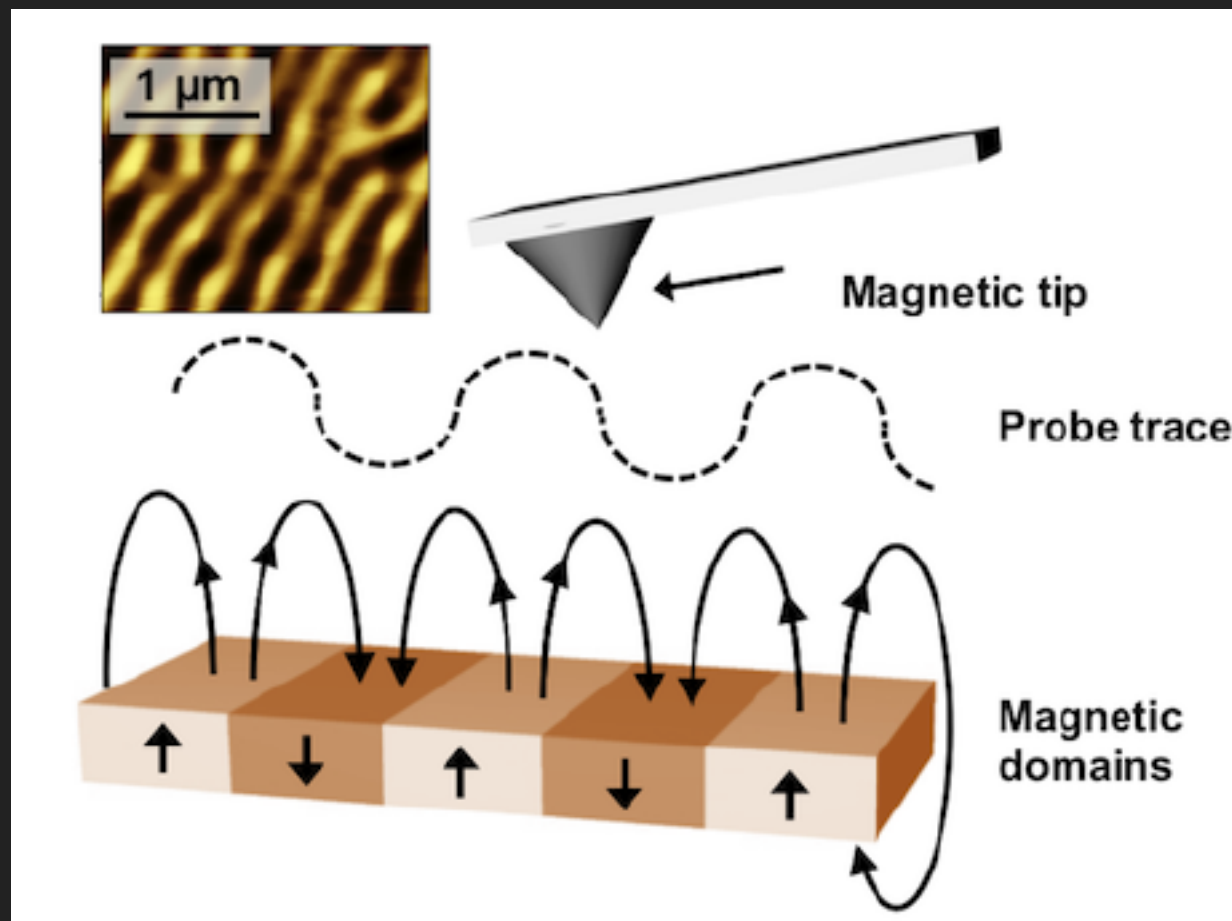
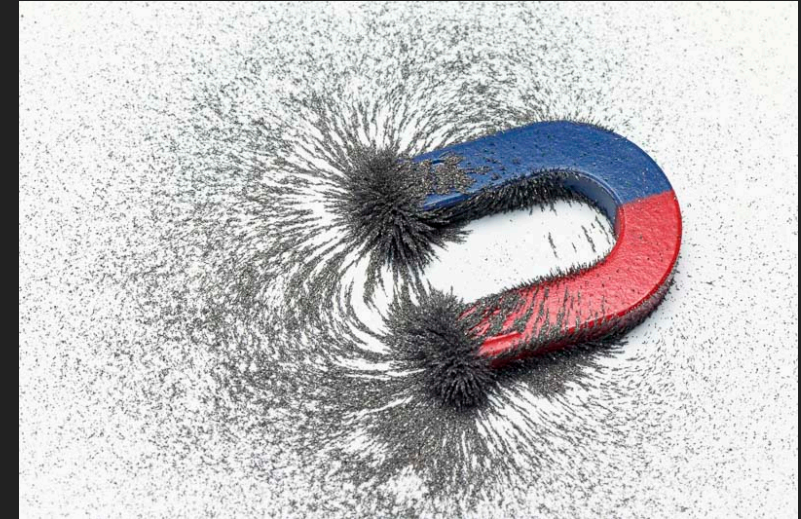
H has a divergence that's not zero! It has sources and sinks associated to the non-uniformity of the magnetisation.



THAT'S WHY A MAGNET WORKS, AFTER ALL

$$\nabla \cdot \vec{H} = -\nabla \cdot \vec{M}$$

H has a divergence that's not zero! It has sources and sinks associated to the non-uniformity of the magnetisation.

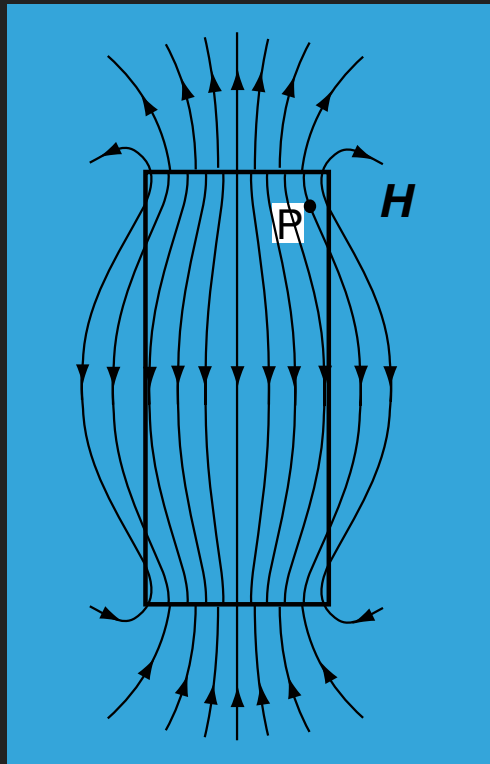


$$\vec{H} = \frac{q_m \vec{r}}{4\pi r^3} \propto \frac{1}{r^2}$$

Fictions charges, the N and S poles do not exist!

However, they can be a convenient mathematical tool to make some kind of calculations.

THE FIELD GENERATED BY A MAGNET



The H-field goes outside the magnet (**fringing field**) but also inside the magnet (**demagnetising field**, notice the opposite direction).

$$\vec{H}_{d,i} = -N_{ij} \vec{M}_j$$

with $i, j = x, y, z$

demagnetising tensor, a symmetric 3x3 matrix with unit trace

DEMAGNETISING FIELD

- ▶ **If the material is an ellipsoid:** its magnetisation is homogeneous.
- ▶ **The demagnetising field** is also homogeneous, and collinear to **M** (although opposing it).
- ▶ **Along the principal axes of the ellipsoid** the three remaining components of the demagnetising tensor are the demagnetising factors N_x, N_y, N_z .

$$N_x + N_y + N_z = 1$$

DEMAGNETISING FACTORS

Shape	M direction	N
Long needle	Parallel to axis	0
	Perpendicular to axis	1/2
Sphere	Any direction	1/3
Thin film	Parallel to plane	0
	Perpendicular to plane	1
Ellipsoid of revolution	$N_c = (1 - 2N_a)$	

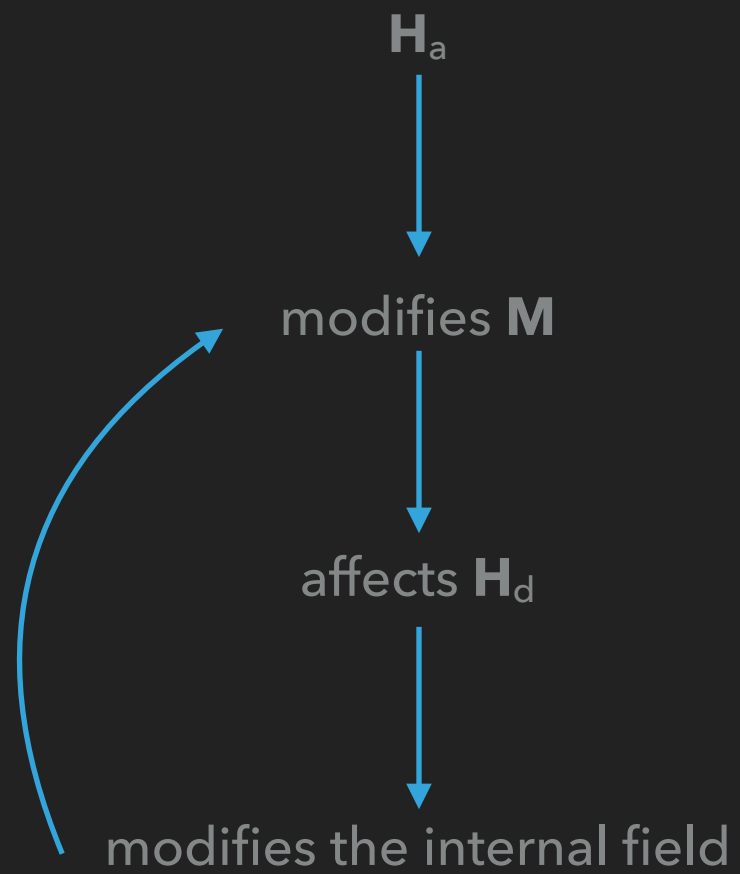
And for other shapes? Complex analytical calculations, approximations, numerical (finite elements) calculations, ...

INTERNAL AND EXTERNAL FIELDS

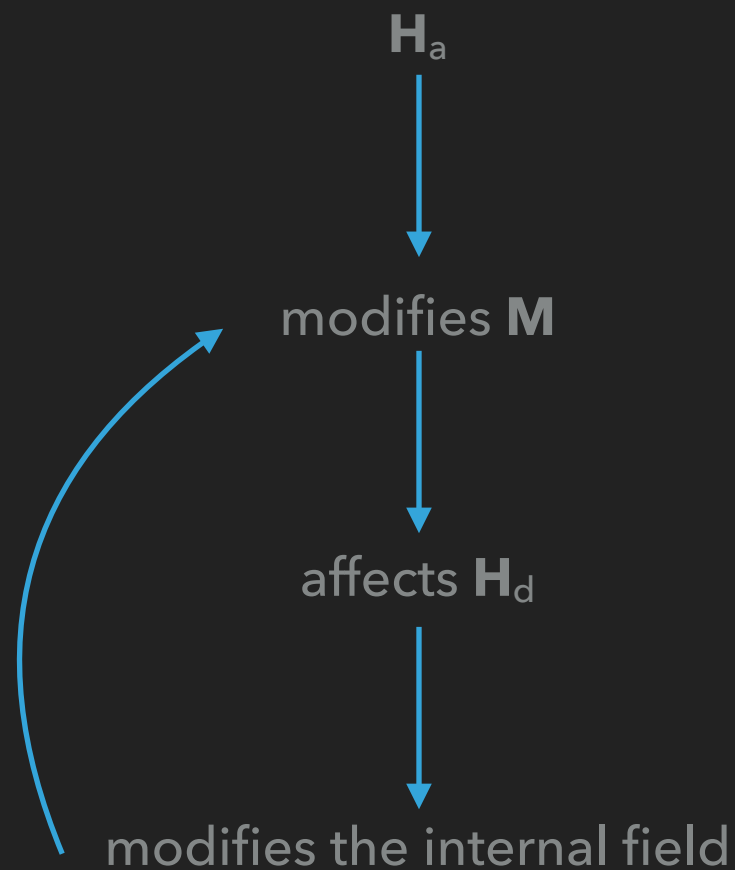
- ▶ **Applied field:** the field acting on a sample produced by electric currents (solenoids, coils, etc.) or by the stray field of magnets (electromagnets, permanent magnets) outside the sample volume. The sample does not contribute to the applied field.
- ▶ **Internal field** in the sample: applied field - demagnetising field produced by the magnetisation of the sample itself:

$$\vec{H} = \vec{H}_a + \vec{H}_d$$

THE DEATH LOOP



THE DEATH LOOP



- closed magnetic circuit measurements (toroids/rings, permeameters, Epstein yokes, etc)
- open magnetic circuit measurements
 - ellipsoidal samples
 - samples with a shape that allows approximate calculation of demagnetising fields
 - samples with standardised shape and size, only H_a is considered, and M is not uniform (an average value is taken as representative)

HYSTERESIS LOOPS

- ▶ $\mathbf{H}$ is taken as the independent variable.
- ▶ $\mathbf{M}$ is measured somehow as a function of $\mathbf{H}$.
- ▶ $\mathbf{B}$ is calculated.
- ▶ If you know $\mathbf{H}_d$: great! You know $\mathbf{H}$.
- ▶ If you do not know $\mathbf{H}_d$ (not even approximately) or don't care (...), use $\mathbf{H}_a$. And then $\mathbf{M}$ is not homogeneous. Learn to live with that.

IN EVERYDAY LIFE

MAGNETIC DIPOLES

- ▶ A magnetised bar submerged into a uniform magnetic field is subject to a **torque**:

$$L = -plH \sin \theta$$

magnetic charges at magnet poles

length of the magnet

angle between **H** and **M**

- ▶ The work done by the torque gives rise to a potential energy **U**:

$$U = -MH \cos \theta$$

M tends to align to **H** (minimum of **U**)

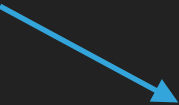
- ▶ When p tends to infinity and l tends to 0, the magnetised bar is a dipole and $m = pl$.

MAGNETIC FIELDS

- ▶ You can only measure external fields.
- ▶ Often you assume that the applied field = external field.
- ▶ Demagnetising fields are among the major responsible for the magnetisation non-uniformity in a magnetic material.
- ▶ Calculating fields outside magnetised materials can be done:
 - ▶ by integrating the dipolar field over the volume distribution of the magnetisation
 - ▶ by replacing the magnetisation by an equivalent distribution of current density (B correct outside and inside the magnetic material)
 - ▶ by replacing the magnetisation by an equivalent distribution of magnetic charges (H correct outside but not inside the magnetic material)

WHY NOT USING MAGNETIC POTENTIALS?

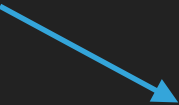
$$\vec{B} = \nabla \times \vec{A}$$



vector potential, $[A] = \text{Tm}$

WHY NOT USING MAGNETIC POTENTIALS?

$$\vec{B} = \nabla \times \vec{A}$$



vector potential, $[A] = \text{Tm}$

Since $\mathbf{B}$ is unchanged by gauge transformations of $\mathbf{A}$:

$$\vec{A} = \frac{1}{2} \vec{B} \times \vec{r}$$

WHY NOT USING MAGNETIC POTENTIALS?

$$\vec{B} = \nabla \times \vec{A}$$

vector potential, $[A] = \text{Tm}$

Since $\mathbf{B}$ is unchanged by gauge transformations of $\mathbf{A}$:

$$\vec{A} = \frac{1}{2} \vec{B} \times \vec{r}$$

For example, the vector potential of a magnetic dipole:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}$$

WHY NOT USING MAGNETIC POTENTIALS?

$$\vec{B} = \nabla \times \vec{A}$$

vector potential, $[A] = \text{Tm}$

Since $\mathbf{B}$ is unchanged by gauge transformations of $\mathbf{A}$:

$$\vec{A} = \frac{1}{2} \vec{B} \times \vec{r}$$

For example, the vector potential of a magnetic dipole:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}$$

Use this on that and you obtain B of a dipole.

WHY NOT USING MAGNETIC POTENTIALS?

$$\vec{H} = -\nabla\varphi_m$$

scalar potential, $[\varphi_m] = \text{A}$

valid only if **H** is produced by magnets
and not by conduction currents

WHY NOT USING MAGNETIC POTENTIALS?

$$\vec{H} = -\nabla\varphi_m$$

scalar potential, $[\varphi_m] = \text{A}$

valid only if $\mathbf{H}$ is produced by magnets
and not by conduction currents

The scalar potential satisfies: $\nabla^2\varphi_m = -\rho_m$

$$\rho_m = -\nabla \cdot \vec{M}$$

magnetic charge density in the bulk

$$\sigma_m = \vec{M} \cdot \vec{e}_n$$

magnetic charge density at the surface

WHY NOT USING MAGNETIC POTENTIALS?

$$\vec{H} = -\nabla\varphi_m$$

scalar potential, $[\varphi_m] = \text{A}$

valid only if $\mathbf{H}$ is produced by magnets
and not by conduction currents

The scalar potential satisfies: $\nabla^2\varphi_m = -\rho_m$

$$\rho_m = -\nabla \cdot \vec{M}$$

magnetic charge density in the bulk

$$\sigma_m = \vec{M} \cdot \vec{e}_n$$

magnetic charge density at the surface

For example a magnetic moment produces a scalar potential of

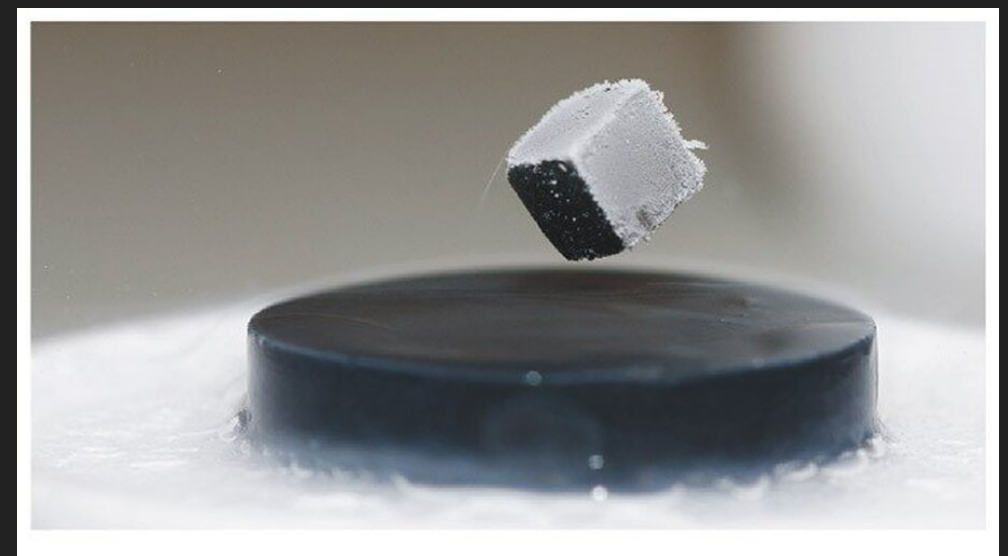
$$\frac{\vec{m} \cdot \vec{r}}{4\pi r^3}$$

AGAIN MAGNETIC DIPOLES

- ▶ A magnetised bar submerged into a non-uniform magnetic field (gradient) is subject to a **force**:

$$\vec{f}_m = \nabla(\vec{m} \cdot \vec{B})$$

- ▶ To minimise the energy, this force pulls the magnet where the field is the greatest (a diamagnet is repelled instead).

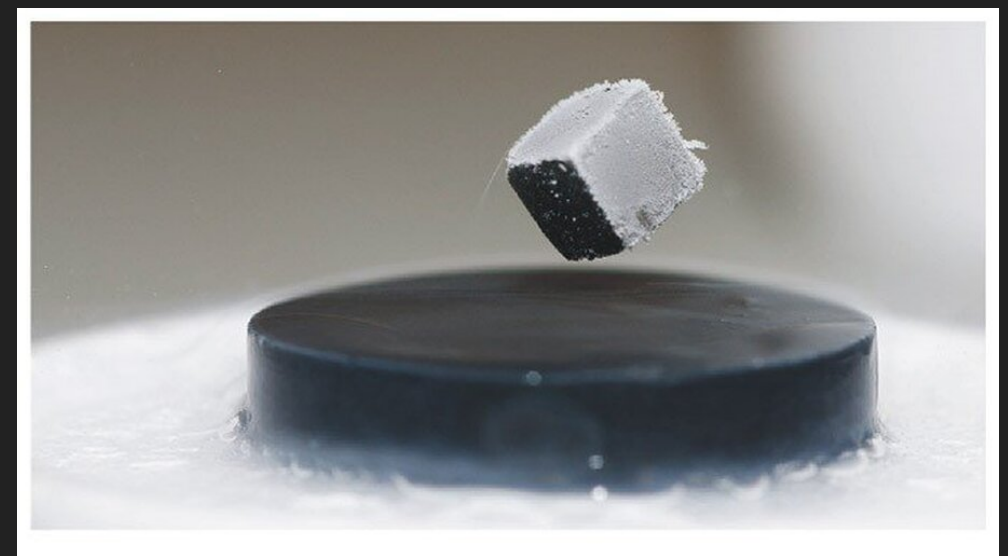
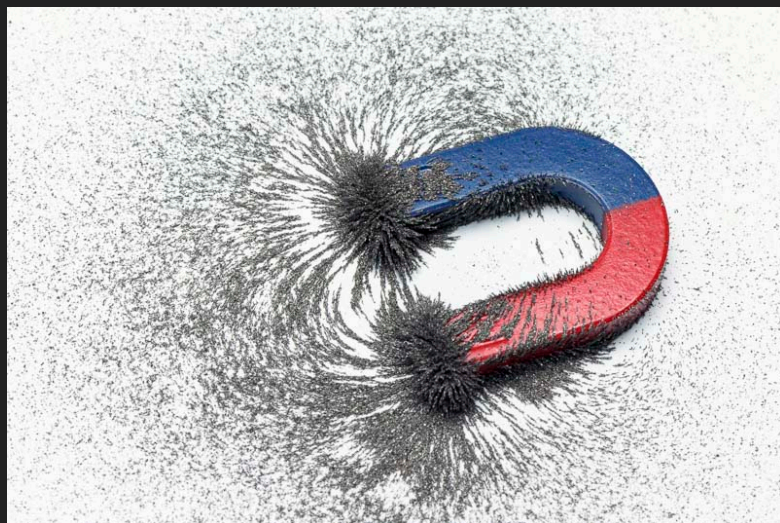


AGAIN MAGNETIC DIPOLES

- ▶ A magnetised bar submerged into a non-uniform magnetic field (gradient) is subject to a **force**:

$$\vec{f}_m = \nabla(\vec{m} \cdot \vec{B})$$

- ▶ To minimise the energy, this force pulls the magnet where the field is the greatest (a diamagnet is repelled instead).



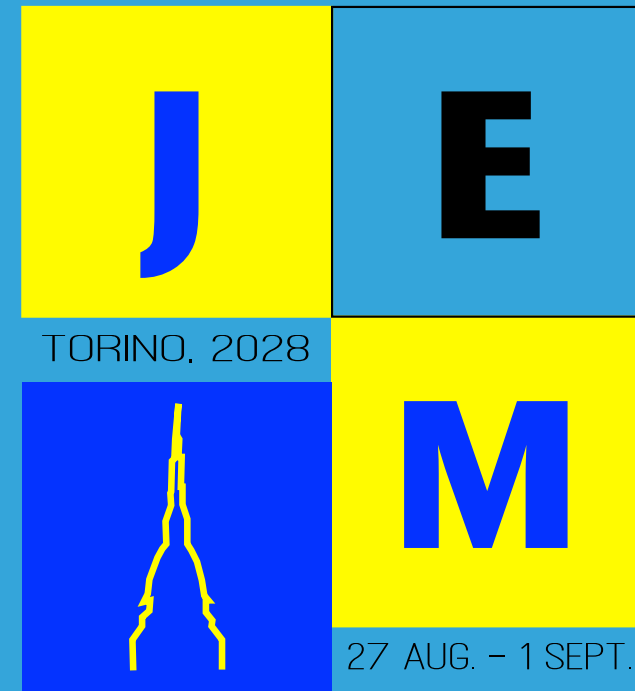
AND DON'T FORGET HYSTERESIS

- ▶ All this said and done, remember that $\mu_r = \mu_r(H)$ through a non-linear, multi-valued function.
- ▶ Hysteresis is:
 - ▶ **non local**: magnetisation at a given point depends on the values of the magnetisation elsewhere.
 - ▶ **rate-dependent**: how fast does the excitation field change in time?
 - ▶ **persistent**: magnetisation at a given field depends on the previous values of the field (to a certain extent).

THAT'S WHY YOU NEED SCHOOLS AND BOOKS

- ▶ **J.M.D. Coey**, *Magnetism and Magnetic Materials*, Cambridge University Press 2009, ISBN-13 978-0-511-67743-4 (eBook) and 978-0-521-81614-4 (hardback).
- ▶ **F. Fiorillo**, *Measurement and Characterisation of Magnetic Materials*, Elsevier Academic Press 2004, ISBN-13 978-0-12-257251-7.
- ▶ **B.D. Cullity, C.D. Graham**, *Introduction to Magnetic Materials (Second Edition)*, Wiley 2009, ISBN-13 978-0-471-47741-9.
- ▶ **G. Bertotti**, *Hysteresis in magnetism*, Academic Press 1998, ISBN 0-12-093270-9.
- ▶ **A. Hubert, R. Schäfer**, *Magnetic Domains*, Springer 1998, ISBN-13 978-3-540-64108-7.

THANK YOU!



m.coisson@inrim.it