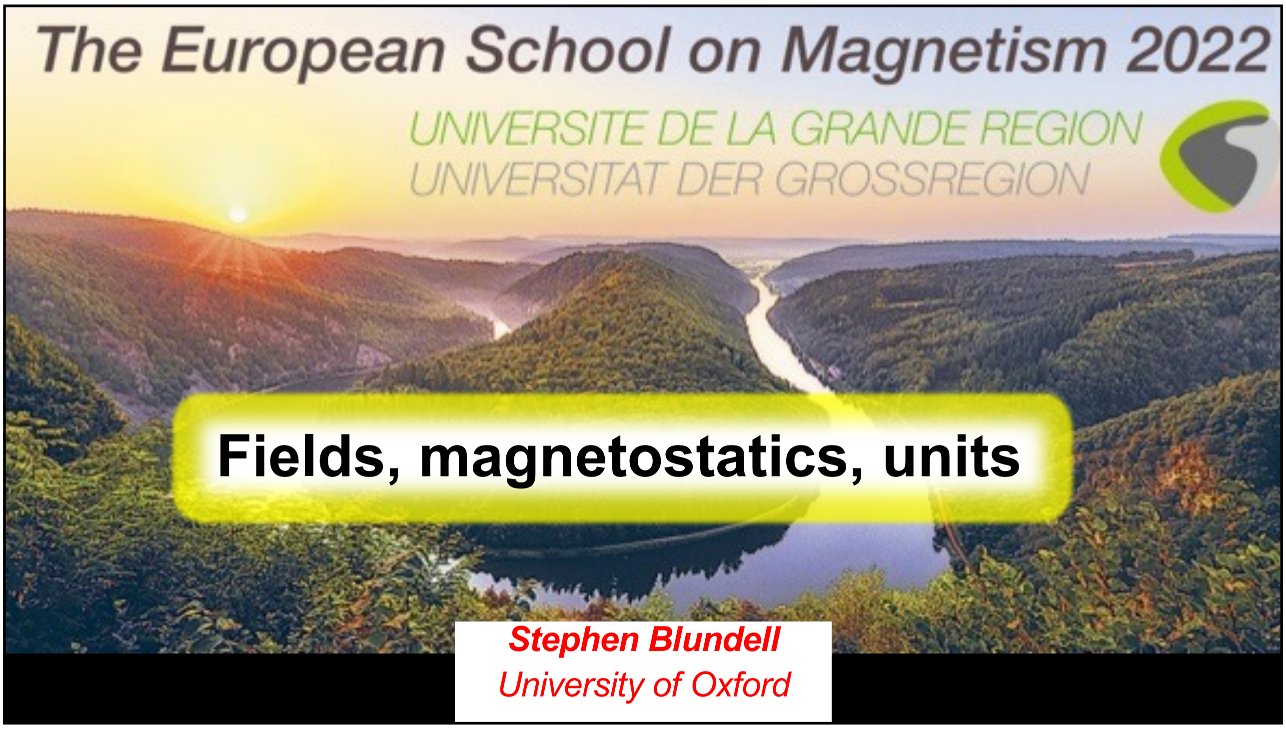

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Fields, magnetostatics, units

Lecture 1

1. Maxwell's equations
2. Poisson's equation and the Biot-Savart Law
3. Multipoles
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Lecture 2

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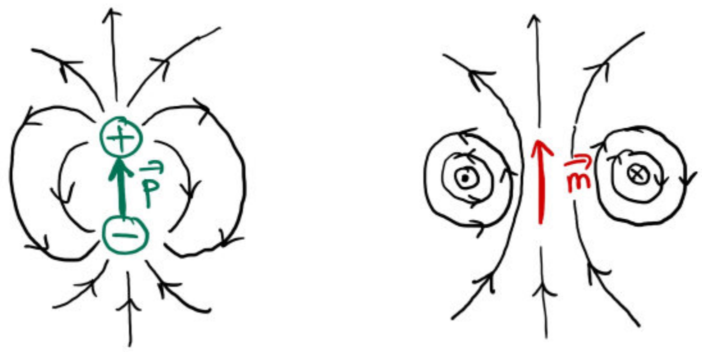
ELECTRIC MAGNETIC



Field patterns look the same at large r
but <u>not</u> at small r.

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Magnetizable materials: in a field $\vec{B}$
acquire a magnetization $\vec{M}$ = <u>magnetic dipole moment per unit volume</u>

e.g. $\vec{M} = n\,\vec{m}$

$\vec{m}$: magnetic moment of one atom/molecule

n: number of magnetic atoms/molecules per unit volume

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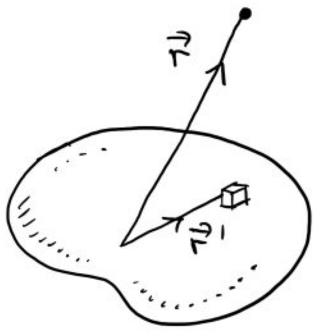
Field due to a box of magnetic dipole moments



$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{M}(\vec{r}')\times(\vec{r}-\vec{r}')\,d\tau'}{|\vec{r}-\vec{r}'|^3}$$

$$= \frac{\mu_0}{4\pi}\int \vec{M}(\vec{r}') \times \nabla'\left(\frac{1}{|\vec{r}-\vec{r}'|}\right)d\tau'$$

$$\nabla'\times\frac{\vec{M}}{|\vec{r}-\vec{r}'|} = \frac{1}{|\vec{r}-\vec{r}'|}\nabla'\times\vec{M} - \vec{M}\times\nabla'\frac{1}{|\vec{r}-\vec{r}'|}$$

hence ...

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$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\nabla'\times\vec{M}(\vec{r}')\,d\tau'}{|\vec{r}-\vec{r}'|} - \underbrace{\frac{\mu_0}{4\pi}\int_V \nabla'\times\frac{\vec{M}(\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\tau'}_{\frac{\mu_0}{4\pi}\oint_S \frac{\vec{M}(\vec{r}')\times d\vec{S}'}{|\vec{r}-\vec{r}'|}}$$

$\int_S \vec{A}\cdot d\vec{S} = \int_\tau \vec{\nabla}\cdot\vec{A}\,d\tau$ Put $\vec{A} = \vec{V}\times\vec{c}$ ← CONSTANT VECTOR

$$\int_S \vec{V}\times\vec{c}\cdot d\vec{S} = -\vec{c}\cdot\int_S \vec{V}\times d\vec{S} \quad ①$$

$$\int_\tau \vec{\nabla}\cdot(\vec{V}\times\vec{c})\,d\tau = \vec{c}\cdot\int\vec{\nabla}\times\vec{V}\,d\tau \quad ②$$
$$-\int\vec{V}\cdot\underbrace{(\vec{\nabla}\times\vec{c})}_{=0}d\tau$$

① = ② and true for any $\vec{c}$ $\Rightarrow$ $\boxed{-\int\vec{V}\times d\vec{S} = \int\nabla\times\vec{V}\,d\tau}$

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$$\Rightarrow \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{J_b(\vec{r}')\,d\tau'}{|\vec{r}-\vec{r}'|} + \frac{\mu_0}{4\pi}\oint \frac{K_b(\vec{r}')\,dS'}{|\vec{r}-\vec{r}'|}$$

where

$\vec{J}_b = \vec{\nabla}\times\vec{M}$ bulk, bound current density

$\vec{K}_b = \vec{M}\times\hat{\vec{n}}$ surface, bound current density

Logic: $\vec{M} \Rightarrow \begin{matrix}\vec{J}_b \\ \vec{K}_b\end{matrix} \Rightarrow \vec{A} \Rightarrow \vec{B}$

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Fix up Ampères law:

$$\vec{\nabla}\times\vec{B} = \mu_0(\vec{J}_f + \underbrace{\vec{J}_b}_{\nabla\times\vec{M}})$$

$$\Rightarrow \vec{\nabla}\times\underbrace{\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right)}_{\text{call this } \vec{H}} = \vec{J}_f$$

$$\Rightarrow \vec{\nabla}\times\vec{H} = \vec{J}_f$$ only depends on free currents

$$\boxed{\vec{B} = \mu_0(\vec{H}+\vec{M})}$$

in cgs: $B = H + 4\pi M$ EXPIRED

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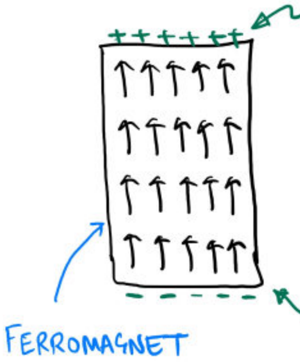
BUT other Maxwell equation is worse.

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \Rightarrow \quad \vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M}$$

$\vec{\nabla}\cdot\vec{M} < 0 \Rightarrow \vec{\nabla}\cdot\vec{H} > 0$



$\vec{\nabla}\cdot\vec{M} > 0 \Rightarrow \vec{\nabla}\cdot\vec{H} < 0$

What about $\vec{J}_b = \nabla\times\vec{M}$?

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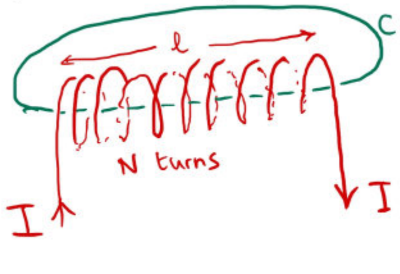
What's the difference between B and H?

Answer #2: Because H is related to free currents by $\vec{\nabla}\times\vec{H} = \vec{J}_f$, it's easy to compute for electromagnets, even with magnetizable materials present



$$\oint_C H\cdot dl = NI$$

$$Hl \approx NI$$

$$H \approx \frac{NI}{l}$$

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Fields, magnetostatics, units

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For linear materials, $\vec{M} \propto \vec{B}$.

Convention is to write $\vec{M} = \chi \vec{H}$

χ: magnetic susceptibility (dimensionless)

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) = \mu_0 (1 + \chi) \vec{H} = \mu \vec{H}$$

μ: permeability

$$\mu = \mu_0 \mu_r = \mu_0 (1 + \chi)$$

μ_r: relative permeability (dimensionless)

Note, materials are only ever linear within certain limits, and sometimes never linear. e.g.

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Fields, magnetostatics, units

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$\chi = \frac{M}{H}$ is dimensionless

EXPIRED (also in cgs, but there often emu cm^{-3})

Also, mass susceptibility $\chi_g = \frac{\chi}{\rho}$ ← DENSITY $(m^3\,kg^{-1})$

and molar susceptibility $\chi_m = \chi V_m$ ← MOLAR VOLUME $(m^3\,mol^{-1})$

EXPIRED (in cgs, χ_m is in emu mol^{-1}
and 1 emu $mol^{-1} = 4\pi \times 10^{-6}\ m^3\,mol^{-1}$.
The 4π comes from $B = H + 4\pi M$
and the 10^{-6} comes from $cm^3 \leftrightarrow m^3$)

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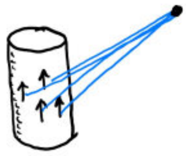
Field calculations

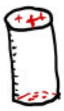
(i) Sum up dipoles $\vec{B} = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}\cdot\hat{\vec{r}})\hat{\vec{r}} - \vec{m}}{r^3}\right]$



(ii) Sum up currents $\vec{J}_b = \vec{\nabla}\times\vec{M}$ and $\vec{K}_b = \vec{M}\times\hat{\vec{n}}$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}_b\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,d^3r' + \frac{\mu_0}{4\pi}\int_S \frac{\vec{K}_b\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,d^2r'$$

(iii) Sum up "charges" $\rho_m = -\nabla\cdot\vec{M}$ $\quad\sigma_m = \vec{M}\cdot\hat{\vec{n}}$



$$\vec{H}(\vec{r}) = \frac{1}{4\pi}\int_V \frac{\rho_m(\vec{r}-\vec{r}\,')\,d^3r'}{|\vec{r}-\vec{r}\,'|^3} + \frac{1}{4\pi}\int \frac{\sigma_m(\vec{r}-\vec{r}\,')\,d^2r'}{|\vec{r}-\vec{r}\,'|^3}$$

Last bit $\because \nabla\times\vec{H}=0 \Rightarrow \vec{H} = -\vec{\nabla}\phi_m$ magnetic scalar potential

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Fields, magnetostatics, units

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What's the difference between B and H?

Answer #3: It's a bit more complicated!

$B_\perp$ is continuous (because $\vec{\nabla}\cdot\vec{B} = 0$)

$H_{//}$ is continuous (because $\vec{\nabla}\times\vec{H} = \vec{J}_f$)

and so what field is inside your sample depends on the <u>sample shape</u>

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Inside a magnet: DEMAGNETIZING FIELD

$$\vec{H}_d = -\underbrace{\overleftrightarrow{N}(\vec{r})}\,\vec{M}$$

↑ demagnetizing tensor (function of position inside the magnet)

Maxwell showed that for an ellipsoid, $\overleftrightarrow{N}(\vec{r}) = \overleftrightarrow{N}$.

Also $\mathrm{Tr}\,\overleftrightarrow{N} = 1$.

So put axes along principal axes of the ellipsoid:

$$\overleftrightarrow{N} = \begin{pmatrix} N_{xx} & 0 & 0 \\ 0 & N_{yy} & 0 \\ 0 & 0 & N_{zz} \end{pmatrix} \quad \text{with } N_{xx}+N_{yy}+N_{zz}=1$$

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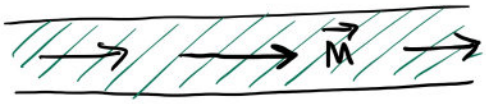
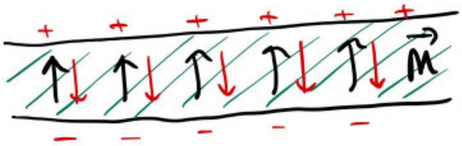
Examples

(i) Thin film (xy plane) $\quad \overleftrightarrow{N} = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$



$\vec{H}_d = 0$



$\vec{H}_d = -\vec{M}$

NB All demagnetizing factors multiply by 4π if using cgs units

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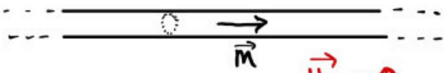
(ii) Sphere

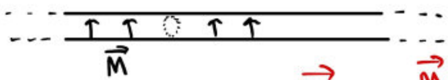
$\overleftrightarrow{N} = \begin{pmatrix} 1/3 & & \\ & 1/3 & \\ & & 1/3 \end{pmatrix}$

$\vec{H}_d = -\dfrac{\vec{M}}{3}$

(iii) long rod // z $\quad \overleftrightarrow{N} = \begin{pmatrix} 1/2 & & \\ & 1/2 & \\ & & 0 \end{pmatrix}$



$\vec{H}_d = 0$



$\vec{H}_d = -\dfrac{\vec{M}}{2}$

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$$\vec{B} = \mu_0(\vec{H} + \vec{M}): \quad \vec{H}_d = -\overleftrightarrow{N}\vec{M} \quad \therefore \quad \vec{B}_d = \mu_0(1-\overleftrightarrow{N})\vec{M}$$

$$\chi_{\text{experimental}} = \frac{M}{H_{\text{applied}}} \qquad \chi_{\text{intrinsic}} = \frac{M}{H_{\text{internal}}}$$

$$H_{\text{internal}} = H_{\text{applied}} - NM$$

$$\Rightarrow \chi_{\text{experimental}} = \frac{M}{H_{\text{internal}} + NM} = \frac{\chi_{\text{intrinsic}}}{1 + N\chi_{\text{intrinsic}}}$$

If $\chi_{\text{intrinsic}} \ll 1$, $\chi_{\text{experimental}} = \chi_{\text{intrinsic}}$

But near T_c, $\chi_{\text{intrinsic}} \to \infty$, $\chi_{\text{experimental}} \to \frac{1}{N}$.

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Calculation of demagnetizing field using Fourier transforms

$$D(\mathbf{r}) = \begin{cases} 1 & \mathbf{r} \text{ inside} \\ 0 & \mathbf{r} \text{ outside} \end{cases}$$

$$\mathbf{M}(\mathbf{r}) = M_0 \hat{\mathbf{m}} D(\mathbf{r})$$

$$\mathbf{B}(\mathbf{k}) = \frac{\mu_0 M_0}{k^2} D(\mathbf{k})(\mathbf{k} \times \hat{\mathbf{m}} \times \mathbf{k})$$

$$B_i = \mu_0(M_i - \mathcal{N}_{ij}M_j)$$

$$\mathcal{N}_{ij}(\mathbf{k}) = D(\mathbf{k})\frac{k_i k_j}{k^2} \qquad \mathrm{Tr}[\mathcal{N}_{ij}(\mathbf{r})] = D(\mathbf{r})$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \mathbf{M}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \mathrm{d}^3\mathbf{r}',$$

$$\mathbf{A}(\mathbf{k}) = \frac{\mu_0}{4\pi}\mathbf{M}(\mathbf{k}) \times \mathcal{F}\left(\frac{\mathbf{r}}{r^3}\right) = -\frac{\mathrm{i}\mu_0}{k^2}\mathbf{M}(\mathbf{k}) \times \mathbf{k}.$$

$$\mathbf{A}(\mathbf{k}) = -\frac{\mathrm{i}B_0}{k^2} D(\mathbf{k})(\hat{\mathbf{m}} \times \mathbf{k}),$$

$$\mathbf{B}(\mathbf{k}) = \mathrm{i}\mathbf{k} \times \mathbf{A}(\mathbf{k}) = \frac{B_0}{k^2} D(\mathbf{k})(\mathbf{k} \times \hat{\mathbf{m}} \times \mathbf{k}),$$

$$\mathbf{B} = \mu_0\mathbf{M} = \frac{B_0}{8\pi^3}\int \mathrm{d}^3\mathbf{k}\, \frac{D(\mathbf{k})}{k^2}\mathbf{k}(\hat{\mathbf{m}} \cdot \mathbf{k}) \exp(\mathrm{i}\mathbf{k} \cdot \mathbf{r}).$$

M.Beleggia, M.DeGraef, JMMM **263,** L1 (2003); M. Beleggia and Y. Zhu, Phil. Mag. **83**, 1045 (2003); M. Beleggia et al. JMMM **278**, 270 (2004).

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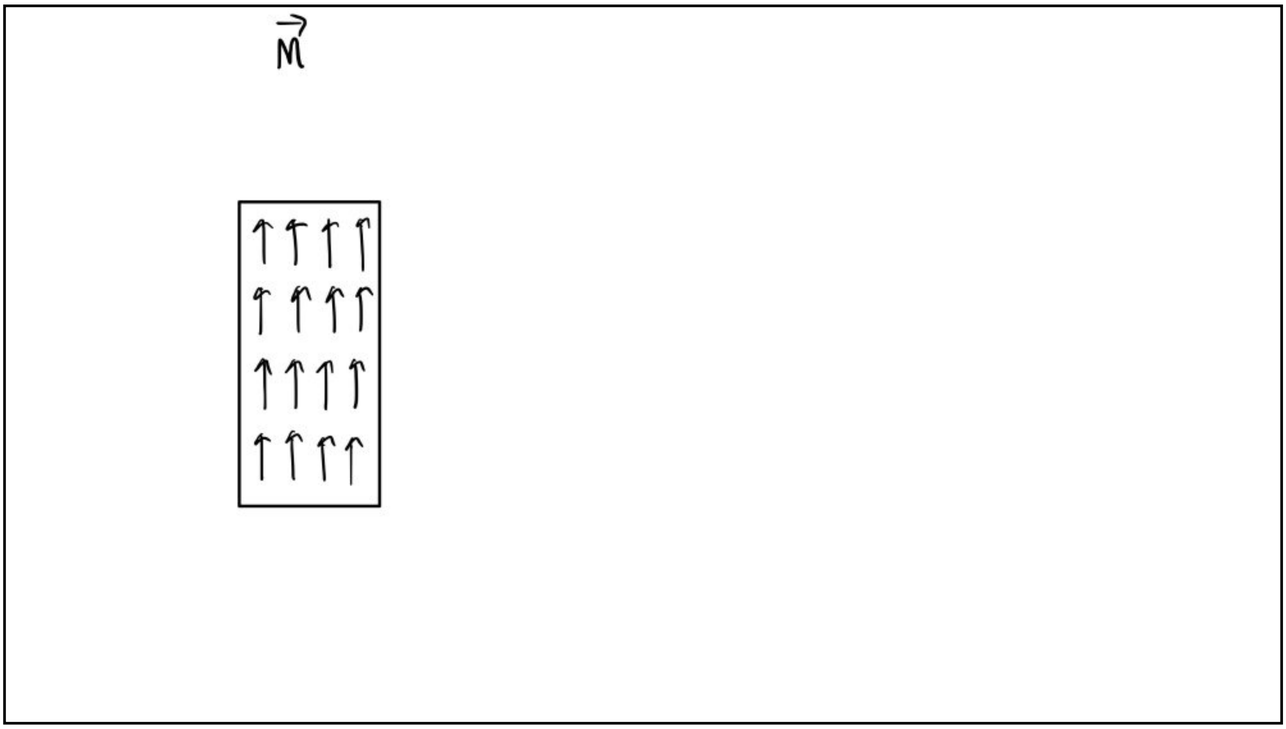

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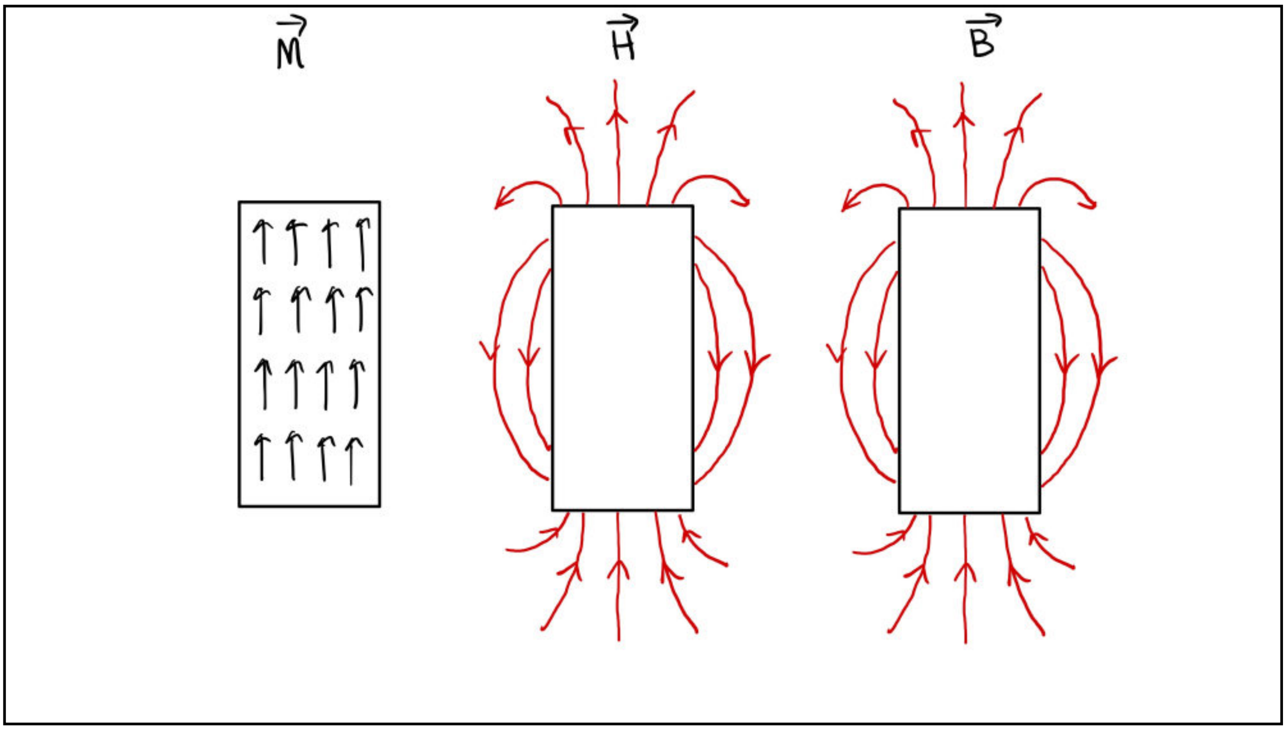

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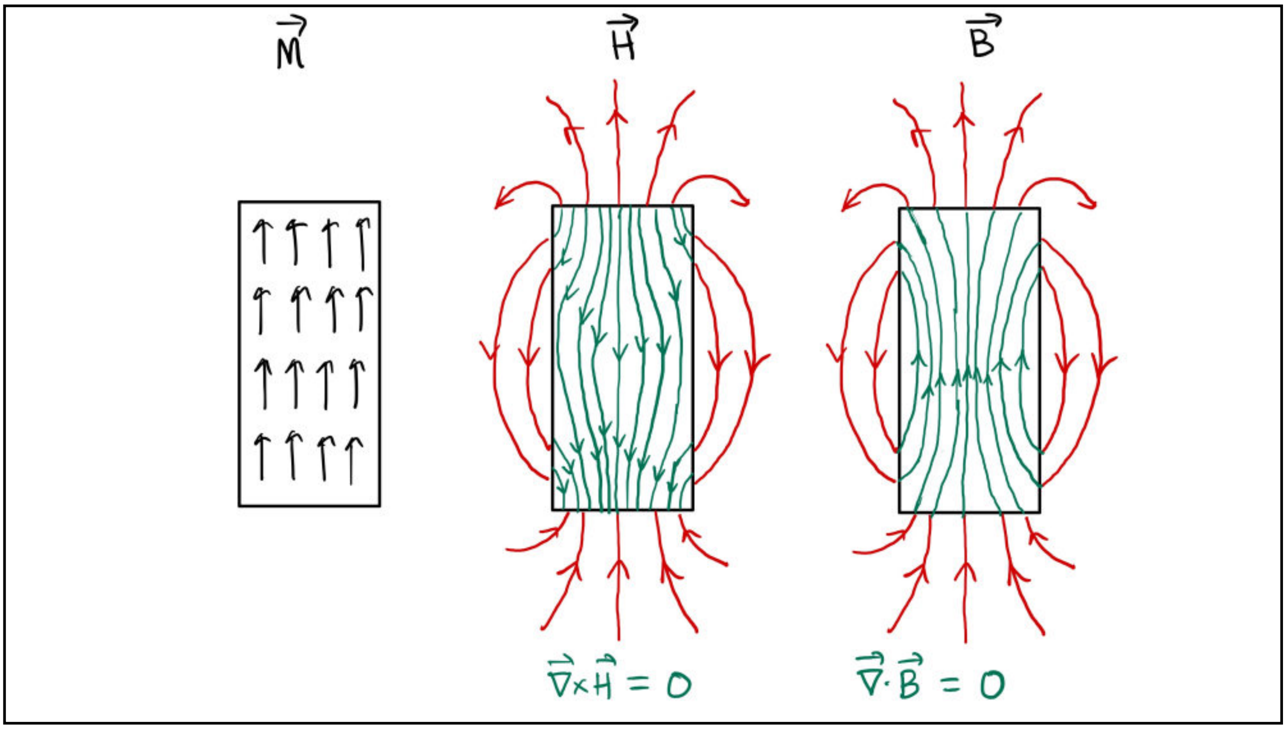

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What's the difference between B and H?

Answer #4 : The two fields are related by
$\vec{B} = \mu_0 (\vec{H} + \vec{M})$ and we need both!

- $\vec{B}$ is more fundamental, and has the "pleasing" divergence equation. $\vec{\nabla} \cdot \vec{B} = 0$ $(\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M})$
- $\vec{H}$ has the nicer curl equation $\vec{\nabla} \times \vec{H} = \vec{J}_f$ $\left(\frac{\vec{\nabla} \times \vec{B}}{\mu_0} = \vec{J}_f + \vec{J}_b\right)$

$\nabla \times \vec{M}$

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Unit conversion (SI to cgs)

Table A.1: Units in the SI system and the cgs system. The abbreviations are m = metre, g = gramme, N = Newton, J = Joule, T = Tesla, G = Gauss, A = Amp, Oe = Oersted, Wb = Weber, Mx = Maxwell. The term emu is short for electromagnetic unit. Note that magnetic susceptibility is dimensionless in SI units.

Quantity	symbol		SI unit		cgs unit
Length	x	10^{-2}	m	$=1$	cm
Mass	m	10^{-3}	kg	$=1$	g
Force	F	10^{-5}	N	$=1$	dyne
Energy	E	10^{-7}	J	$=1$	erg
Magnetic induction	$\mathbf{B}$	10^{-4}	T	$=1$	G
Magnetic field strength	$\mathbf{H}$	$10^3/4\pi$	$\mathrm{A\,m^{-1}}$	$=1$	Oe
Magnetic moment	$\boldsymbol{\mu}$	10^{-3}	$\mathrm{J\,T^{-1}}$ or $\mathrm{A\,m^2}$	$=1$	$\mathrm{erg\,G^{-1}}$
Magnetization (= moment per volume)	$\mathbf{M}$	10^3	$\mathrm{A\,m^{-1}}$ or $\mathrm{J\,T^{-1}\,m^{-3}}$	$=1$	Oe
Magnetic susceptibility	χ	4π		$=1$	$\mathrm{emu\,cm^{-3}}$
Molar susceptibility	χ_{m}	$4\pi\times10^{-6}$	$\mathrm{m^3\,mol^{-1}}$	$=1$	$\mathrm{emu\,mol^{-1}}$
Mass susceptibility	χ_{g}	$4\pi\times10^{-3}$	$\mathrm{m^3\,kg^{-1}}$	$=1$	$\mathrm{emu\,g^{-1}}$
Magnetic flux	ϕ	10^{-8}	$\mathrm{Tm^2}$ or Wb	$=1$	$\mathrm{G\,cm^{-2}}$ or Mx
Demagnetization factor	N		$0<\mathrm{N}<1$		$0<\mathrm{N}<4\pi$

EXPIRED

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