The European School on Magnetism 2019 – Practical on domain walls – Answers

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1 Simple description

1.1 Dimensional analysis

The only physical parameters in the system are K_u and A. K_u has unit $\mathrm{J/m^3}$, while A has unit $\mathrm{J/m}$. The only way to exhibit a length with these is with the quantity $\delta = \sqrt{A/K_\mathrm{u}}$. The only way to exhibit an energy with these is with the quantity $\mathcal{E} = \sqrt{AK_\mathrm{u}}$. From these, it is clear that K_u tends to shrink the wall width, while A tends to increase it. Both contribute to the wall energy.

1.2 Simple variational model

After integration of the local energy, one finds: $\mathcal{E} = K_\mathrm{u}\ell/2 + A\pi^2/\ell$. The minimization yields $\ell_\mathrm{var} = \pi\sqrt{2}\sqrt{A/K_\mathrm{u}}$ and $\mathcal{E}_\mathrm{var} = \pi\sqrt{2}\sqrt{AK_\mathrm{u}}$ is the associated energy.

2 Euler-Lagrange equation

The local minima of a function $f(x)$ are determined by the cancellation of its first derivative $\mathrm{d}f/\mathrm{d}x$, combined with a positive curvature $\mathrm{d}^2f/\mathrm{d}x^2 > 0$ at the same location. The situation is more complex here as we are not considering a single degree of freedom x, however an infinite number of degrees of freedom, *i.e.* the values of $\theta(x)$ and $\mathrm{d}\theta/\mathrm{d}x$ at each location x. The purpose of the Euler-Lagrange formalism is to boil down to a local problem.

In analogy with the search for a local minimum for a usual function $f(x)$, we will consider the infinitesimal variation $\delta\mathcal{E}$ to $\mathcal{E}$, induced by an infinitesimal variation $\delta\theta(x)$ of the function $\theta(x)$. Note that $\delta(\mathrm{d}\theta/\mathrm{d}x) = \mathrm{d}(\delta\theta)/\mathrm{d}x$. We will consider that an extremum of $\mathcal{E}$ is found, when $\delta\mathcal{E}$ does not depend on $\delta\theta$, to first order.

To do this, let us proceed to the first-order expansion of the quantities considered in Eq.(2) from the main text. The function E is formally considered as depending on two variables α and β, and reads, in the case of a micromagnetic problem:

$$E(\alpha,\beta) = A\beta^2 + E_\mathrm{a}(\alpha) \tag{1}$$

The first-order expansion for an infinitesimal variation reads:

$$\delta E(\alpha,\beta) = \frac{\partial E}{\partial\alpha}\delta\alpha + \frac{\partial E}{\partial\beta}\delta\beta \tag{2}$$

Based on this, let us come back to the case where $\alpha = \theta(x)$ and $\beta = \mathrm{d}\theta/\mathrm{d}x(x)$:

$$\mathcal{E}[\theta+\delta\theta] = \mathcal{E}[\theta] + \int_{x_A}^{x_B}\left[\frac{\partial E}{\partial\frac{\mathrm{d}\theta}{\mathrm{d}x}}\delta\left[\frac{\mathrm{d}\theta}{\mathrm{d}x}(x)\right] + \frac{\partial E}{\partial\theta}\delta\theta(x)\right]\mathrm{d}x + \frac{\mathrm{d}\mathcal{E}_A}{\mathrm{d}\theta}\left[\theta(x_A)\right]\delta\theta(x_A) + \frac{\mathrm{d}\mathcal{E}_B}{\mathrm{d}\theta}\left[\theta(x_B)\right]\delta\theta(x_B). \tag{3}$$

The integrant with the $\delta\theta(x)$ term is fine, because ultimately we wish to get an expression of $\delta\mathcal{E}$ versus this quantity. Let us work on the first quantity, based on integration by parts:

$$\begin{aligned}\int_{x_A}^{x_B}\frac{\partial E}{\partial\frac{\mathrm{d}\theta}{\mathrm{d}x}}\left[\frac{\mathrm{d}\delta\theta}{\mathrm{d}x}(x)\right]\mathrm{d}x &= \left[\frac{\partial E}{\partial\frac{\mathrm{d}\theta}{\mathrm{d}x}}\delta\theta\right]_{x_A}^{x_B}-\int_{x_A}^{x_B}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial E}{\partial\frac{\mathrm{d}\theta}{\mathrm{d}x}}\right)(x)\delta\theta(x)\mathrm{d}x\\ &= \left.\frac{\partial E}{\partial\frac{\mathrm{d}\theta}{\mathrm{d}x}}\right|_B\delta\theta(x_B)-\left.\frac{\partial E}{\partial\frac{\mathrm{d}\theta}{\mathrm{d}x}}\right|_A\delta\theta(x_A)-\int_{x_A}^{x_B}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial E}{\partial\frac{\mathrm{d}\theta}{\mathrm{d}x}}\right)(x)\delta\theta(x)\mathrm{d}x \qquad (4)\end{aligned}$$

Turning back to Eq.(3), the variation $\delta\mathcal{E}$ is zero to first order if the coefficients in front of $\delta\theta(x_A)$ and $\delta\theta(x_B)$ are zero, as well as the functional coefficient to $\delta\theta(x)$ is identically zero. This leads to Eqs.(3-5) from the main text.

3 Micromagnetic Euler equation

Eq.(3-5) from the main text read, written for a micromagnetic case:

$$\frac{\partial E_\mathrm{a}}{\partial\theta}-2A\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} \equiv 0 \qquad (5)$$

$$\frac{\mathrm{d}\mathcal{E}_A}{\mathrm{d}\theta}(x_A)-2A\frac{\mathrm{d}\theta}{\mathrm{d}x}(x_A) = 0 \qquad (6)$$

$$\frac{\mathrm{d}\mathcal{E}_B}{\mathrm{d}\theta}(x_B)+2A\frac{\mathrm{d}\theta}{\mathrm{d}x}(x_B) = 0 \qquad (7)$$

Eq.(5) is the differential equation requested. In each of these equations, the left term is the first derivative of the energy against the degree of freedom. It is thus the torque exerted on magnetization by microscopic terms, *e.g.* magnetic anisotropy. The right term must also be a torque, obviously arising from exchange. The term in the surface equations is the easiest to understand. If we think of the very last magnetic moment present at the interface (A or B), it undergoes a torque that tends to align it with its neighbor. The value of the sign is clear as well: bringing it downwards at A if $\mathrm{d}\theta/\mathrm{d}x>0$, and upwards at B if $\mathrm{d}\theta/\mathrm{d}x>0$. The exchange torque in the first equation can be understood in the following manner: it results from the opposite torques exerted at the left and right sides of a given moment. Performing this difference is equivalent to performing a derivation, which explains that it is a second order derivative appearing in the equation.

Let us proceed to the integration of Eq.(5). It can be transformed into:

$$\frac{\partial E_\mathrm{a}}{\partial\theta}\frac{\mathrm{d}\theta}{\mathrm{d}x}=2A\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2}\frac{\mathrm{d}\theta}{\mathrm{d}x} \qquad (8)$$

which itself is:

$$\frac{\mathrm{d}E_\mathrm{a}}{\mathrm{d}x}=A\frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2\right] \qquad (9)$$

if E_a is considered now a function of space. Upon integration from *e.g.* the A boundary, one gets:

$$E_\mathrm{a}(x)-E_\mathrm{a}(x_A)=A\left[\frac{\mathrm{d}\theta}{\mathrm{d}x}(x)\right]^2-A\left[\frac{\mathrm{d}\theta}{\mathrm{d}x}(x_A)\right]^2 \qquad (10)$$

Extended free boundary conditions means that A lies clearly inside a domain, so that $\mathrm{d}\theta/\mathrm{d}x(x_A)=0$, and magnetization is aligned along an easy axis, with the minimum energy. Let us define that this minimum to be zero, which is also possible: $E_\mathrm{a}(A)=0$. It is also always possible to make a choice of angles such that $\frac{\mathrm{d}\theta}{\mathrm{d}x}>0$, so that the above becomes:

$$E_\mathrm{a}(x)=A\left(\frac{\mathrm{d}\theta}{\mathrm{d}x}\right)^2 \qquad (11)$$

The above means equipartition of anisotropy and exchange energy at every location in the domain wall. This is a key result, stemming from the quadratic form for the exchange. This will allow us to derive easily the total energy of the domain wall. From Eq.(1) in the main text, and replacing each term by the geometrical mean of the two terms (as they are equal):

$$\mathcal{E}[\theta] = 2\int_{x_A}^{x_B} \sqrt{AE_\mathrm{a}(\theta)}\,\frac{\mathrm{d}\theta}{\mathrm{d}x}\,\mathrm{d}x \tag{12}$$

$$= 2\int_{\theta(x_A)}^{\theta(x_B)} \sqrt{AE_\mathrm{a}(\theta)}\,\mathrm{d}\theta \tag{13}$$

This method of determining the wall energy is important in analytical micromagnetic modelling. Indeed, not only does it provide a simple result: it is sometimes not possible to solve analytically the wall profile $\theta(x)$, yet its total energy can be calculated.

4 The Bloch domain wall

The domain wall width δ is a length, which may depend on the only two material parameters at play in the situation: exchange stiffness A expressed in J/m, and anisotropy coefficient K_u expressed in J/m^3. The only way to build a length out of these is $\Delta_\mathrm{u} = \sqrt{A/K_\mathrm{u}}$, with which δ is expected to scale. The domain wall energy $\mathcal{E}$ has units J/m^2. Similarly to the above, we must have: $\mathcal{E} \sim \sqrt{AK_\mathrm{u}}$. Variables in Eq.(11) can be split, to allow integration:

$$\mathrm{d}x = \Delta_\mathrm{u}\frac{\mathrm{d}\theta}{\sin\theta} \tag{14}$$

as $\sin\theta \geq 0$. This can be integrated noticing $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$, and that the derivative of $\tan\theta$ is $1/\cos^2\theta$.

As regards the wall width, $\delta_\mathrm{Bl} = \pi(\mathrm{d}x/\mathrm{d}\theta)(\theta = \pi/2)$ from the asymptote. Similar to the calculation of the wall energy, it would be cumbersome to come back to the exact wall profile $\theta(x)$. Using Eq.(14), one gets immediately: $\delta_\mathrm{Bl} = \pi\Delta_\mathrm{u}$.