A Question

What is Magnetism ?

Basic Concepts: Magnetostatics

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1. Introduction
2. Magnets
3. Fields
4. Forces and energy
5. Units and dimensions

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This series of three lectures covers basic concepts in magnetism; Firstly magnetic moment, magnetization and the two magnetic fields are presented. Internal and external fields are distinguished. The main characteristics of ferromagnetic materials are briefly introduced. Magnetic energy and forces are discussed. SI units are explained, and dimensions are given for magnetic, electrical and other physical properties.

Then the electronic origin of paramagnetism of non-interacting electrons is calculated in the localized and delocalized limits. The multi-electron atom is analysed, and the influence of the local crystalline environment on its paramagnetism is explained.

Assumed is an elementary knowledge of solid state physics, electromagnetism and quantum mechanics.

Books

Some useful books include:

- J. M. D. Coey; *Magnetism and Magnetic Magnetic Materials*. Cambridge University Press (2010) 614 pp
An up to date, comprehensive general text on magnetism. Indispensable!

- Stephen Blundell *Magnetism in Condensed Matter*, Oxford 2001
A good treatment of the basics.

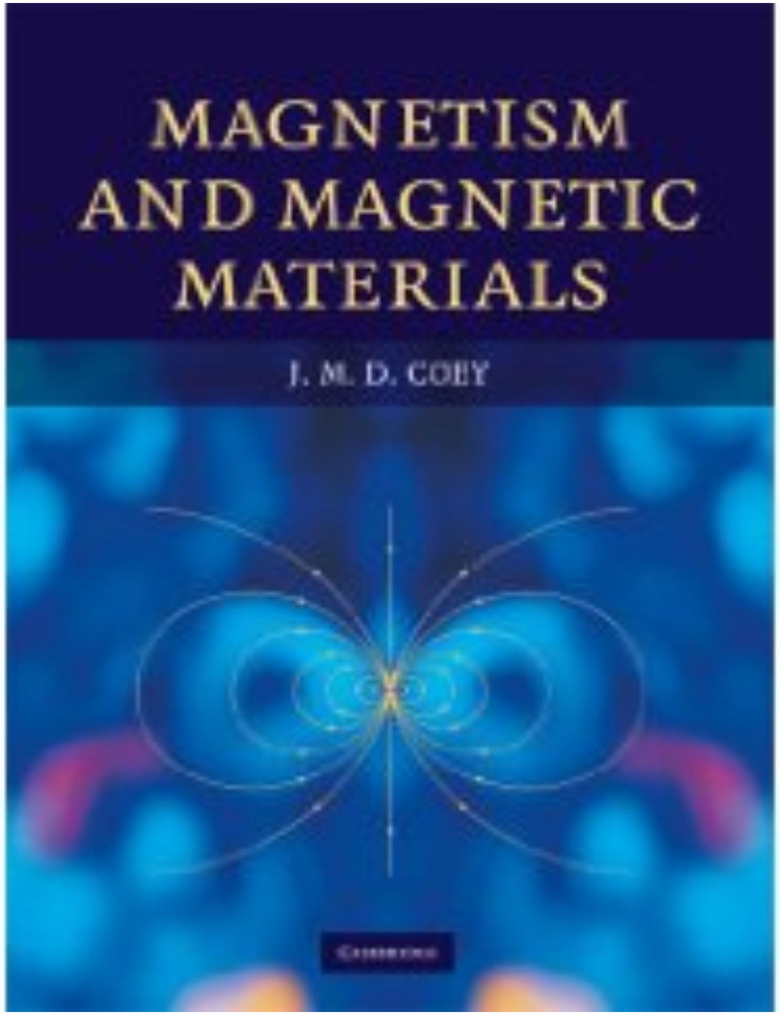
- D. C. Jilles *An Introduction to Magnetism and Magnetic Magnetic Materials*, *Magnetic Sensors and Magnetometers*, CRC Press 480 pp

- J. D. Jackson *Classical Electrodynamics* 3rd ed, Wiley, New York 1998
A classic textbook, written in SI units.

- G. Bertotti, *Hysteresis* Academic Press, San Diego 2000
A monograph on magnetostatics

- A. Rosencwaig *Ferrohydrodynamics*, Dover, Mineola 1997
A good account of magnetic energy and forces

- L.D. Landau and E. M. Lifschitz *Electrodynamics of Continuous Media* Pergammon, Oxford 1989
The definitive text



614 pages. Published March 2010

Available from Amazon.co.uk ~€50

www.cambridge.org/9780521816144

1 Introduction

2 Magnetostatics

3 Magnetism of the electron

4 The many-electron atom

5 Ferromagnetism

6 Antiferromagnetism and other magnetic order

7 Micromagnetism

8 Nanoscale magnetism

9 Magnetic resonance

10 Experimental methods

11 Magnetic materials

12 Soft magnets

13 Hard magnets

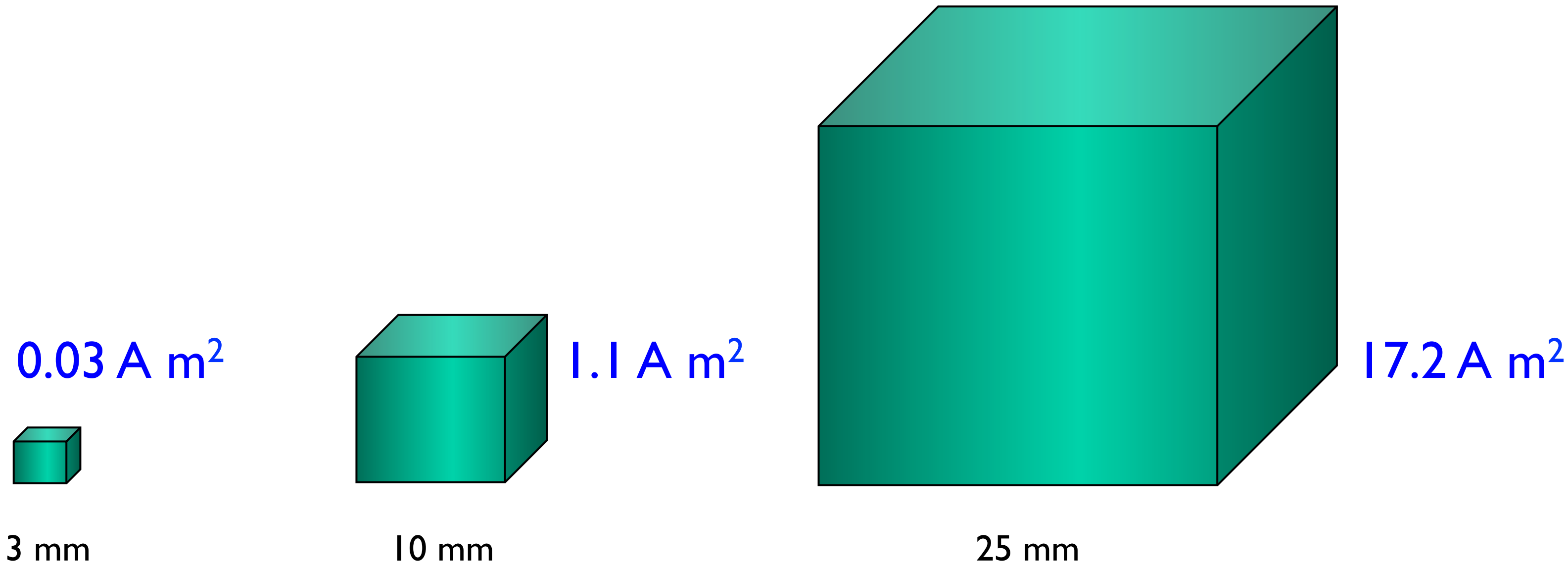
14 Spin electronics and magnetic recording

15 Other topics

Appendices, conversion tables.

1. Introduction

Magnets and magnetization



m is the magnetic (dipole) moment of the magnet. It is proportional to volume

$$\boxed{\boldsymbol{m} = \boldsymbol{M}V}$$

magnetic moment

magnetization

volume

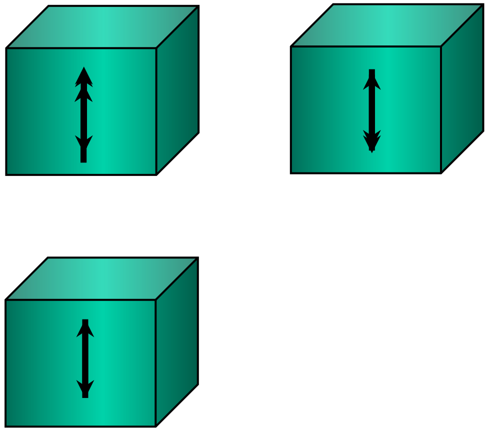
Suppose they are made of $Nd_2Fe_{14}B$
($M \approx 1.1$ MA m^{-1})

What are the moments?

Magnetization is the intrinsic property of the material;
Magnetic moment is a property of a particular magnet.

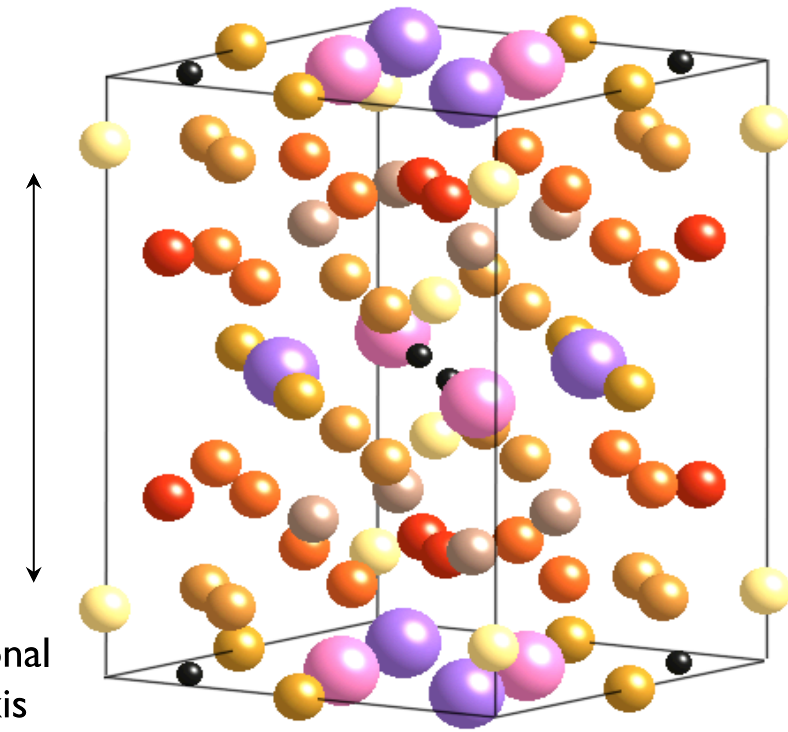
Magnetic moment - a vector

Each magnet creates a field around it. This acts on *any* material in the vicinity but strongly with another magnet. The magnets attract or repel depending on their mutual orientation



↑ ↑	Weak repulsion
↑ ↓	Weak attraction
← ←	Strong attraction
← →	Strong repulsion

$Nd_2Fe_{14}B$



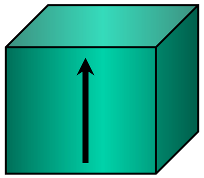
tetragonal easy axis

Units

What do the units mean?

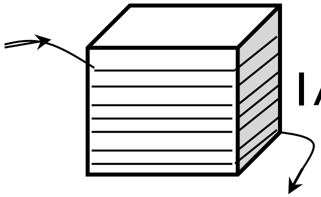
m – A m^2

M – A m^{-1}



1 A m^2

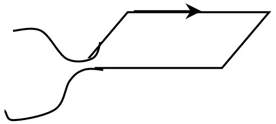
$\equiv$



10,000 turns

1A

$\equiv$



10,000 A

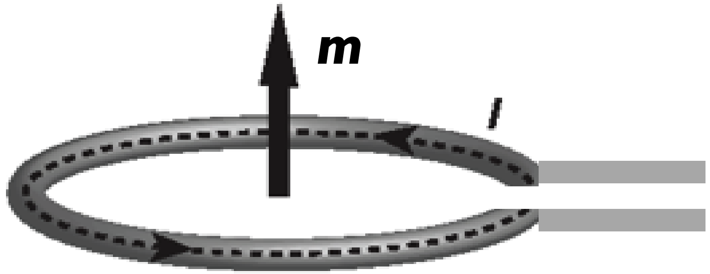
Permanent magnets win over electro-magnets at small sizes

$\boxed{\boldsymbol{m} = I\mathcal{A}}$

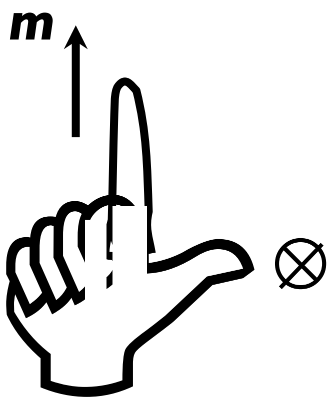
area of the loop

$\boxed{\boldsymbol{m} = nI\mathcal{A}}$

number of turns

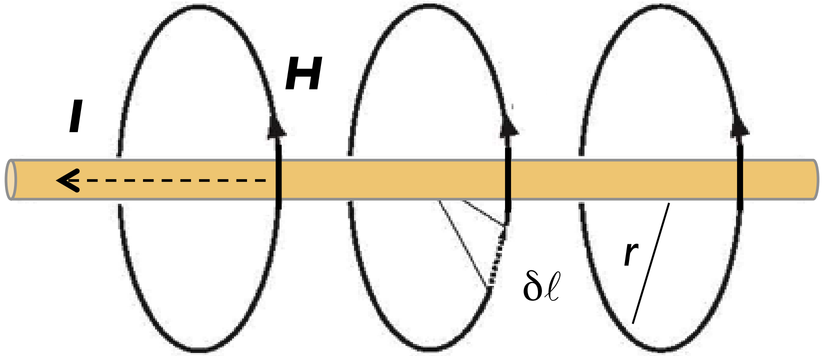


Ampère, 1821. A current loop or coil is equivalent to a magnet

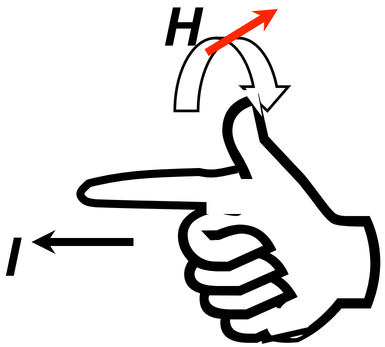


Right-hand corkscrew

Magnetic field ***H*** – Oersted's discovery



The relation between electric current and magnetic field was discovered by Hans-Christian Øersted, 1820.



Right-hand corkscrew

$$\boxed{\oint H \mathrm{d}\ell = I}$$

Ampère's law

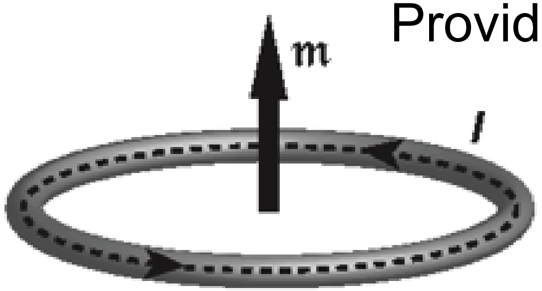
$H = I/2\pi r$

If I = 1 A, r = 1 mm
H = 159 A m^{-1}

Earth's field ≈ 40 Am^{-1}

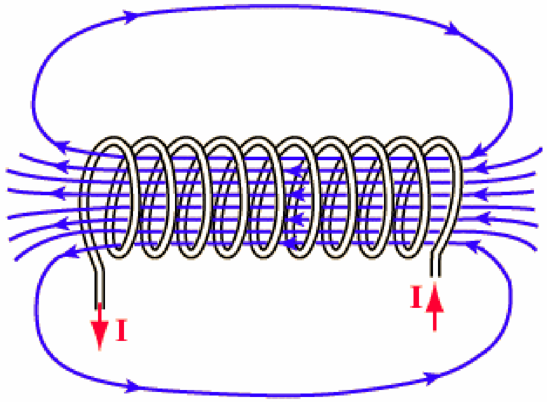
Magnets and currents – Ampere and Arago's insight

A magnetic moment is equivalent to a current loop.



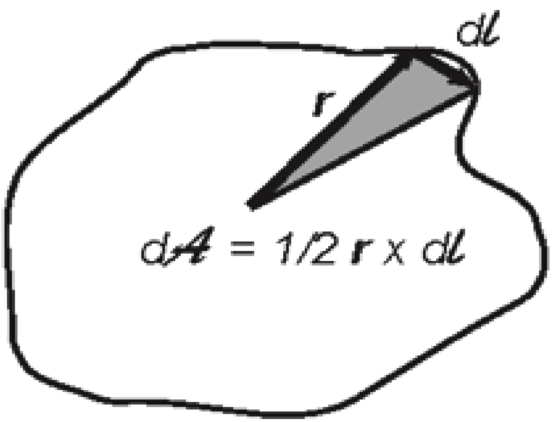
Provided the current flows in a plane

$$\boxed{\boldsymbol{m} = I\mathcal{A}}$$



long solenoid with n turns. $\boldsymbol{m} = nI\mathcal{A}$

In general:



$$\boxed{\boldsymbol{m} = \tfrac{1}{2}\int \boldsymbol{r} \times \boldsymbol{j}(\boldsymbol{r})\, d^3r}$$

Where j is the current density;

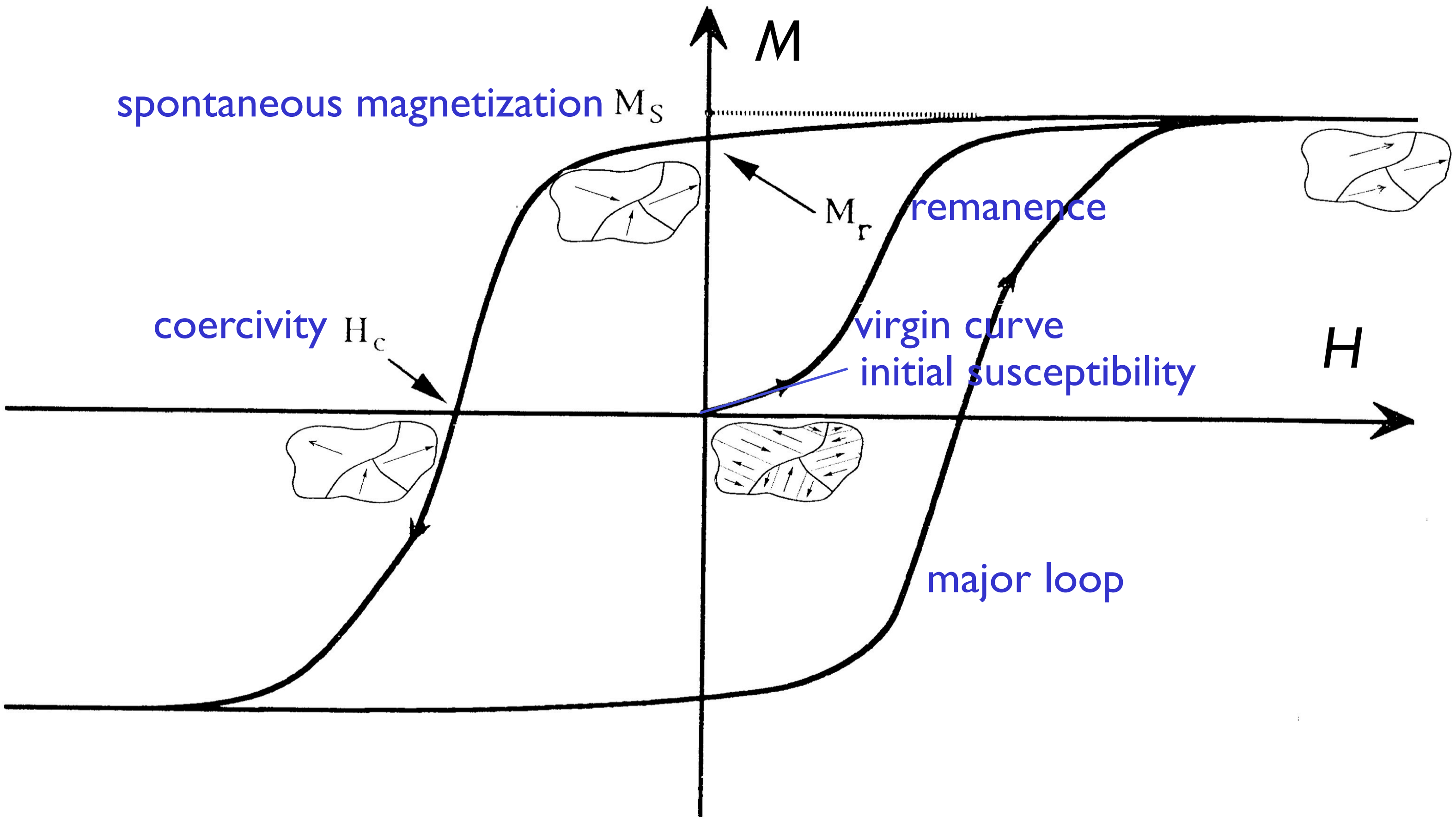
$I = \boldsymbol{j}.\boldsymbol{a}$

So $\boldsymbol{m} = \tfrac{1}{2}\int \boldsymbol{r} \times I d\boldsymbol{l} = I\int d\mathcal{A} = \boldsymbol{m}$

$\boldsymbol{j}_m = \nabla \times \boldsymbol{M}$

	Space inversion
Polar vector	$-\boldsymbol{j}$
Axial vector	$\boldsymbol{m}$

Magnetization curves - Hysteresis loop



The hysteresis loop shows the irreversible, nonlinear response of a ferromagnet to a magnetic field . It reflects the arrangement of the magnetization in ferromagnetic *domains*. A broad loop like this is typical of a *hard* or *permanent* magnet.

What is Magnetostatics ?

Maxwell's Equations

$$\nabla \cdot \boldsymbol{D} = \rho,$$

$$\nabla \cdot \boldsymbol{B} = 0,$$

$$\nabla \times \boldsymbol{E} = \cancel{-\partial \boldsymbol{B}/\partial t},$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{j} + \cancel{\partial \boldsymbol{D}/\partial t}.$$

Electromagnetism with no time-dependence

In magnetostatics, we have only magnetic material and circulating currents in conductors, all in a steady state. The fields are produced by the magnets & the currents

$$\boxed{\nabla \cdot \boldsymbol{j} = 0 \quad \nabla \cdot \boldsymbol{B} = 0 \quad \nabla \times \boldsymbol{H} = \boldsymbol{j}.}$$

2. Magnetization

Magnetic Moment and Magnetization

The magnetic moment $\boldsymbol{m}$ is the elementary quantity in solid state magnetism.

Define a local moment density - magnetization – $\boldsymbol{M}(\boldsymbol{r}, t)$ which fluctuates wildly on a sub-nanometer and a sub-nanosecond scale.
Define a mesoscopic average magnetization

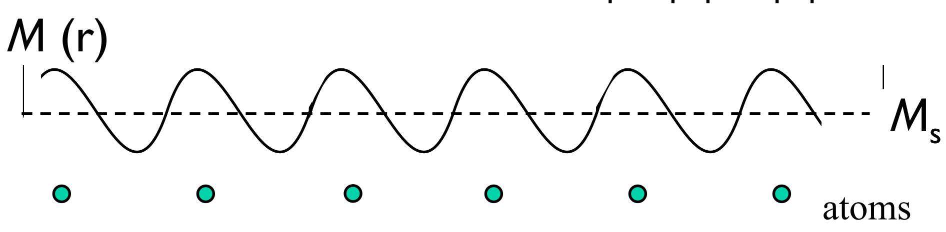
$$\boxed{\delta \boldsymbol{m} = \boldsymbol{M} \delta V} \qquad \boldsymbol{M} = \delta \boldsymbol{m} / \delta V$$

The continuous medium approximation

$\boldsymbol{M}$ can be the spontaneous magnetization $\boldsymbol{M}_s$ within a ferromagnetic domain

A macroscopic average magnetization is the domain average

$$\boldsymbol{M} = \Sigma_i \boldsymbol{M}_i V_i / \Sigma_i V_i$$



The mesoscopic average magnetization

Magnetization and current density

The magnetization of a solid is somehow related to a 'magnetization current density' $\boldsymbol{J}_m$ that produces it.

Since the magnetization is created by bound currents, $\int_s \boldsymbol{J}_m . d\mathcal{A} = 0$ over any surface.

Using Stokes theorem $\oint \boldsymbol{M}.d\ell = \int_s (\nabla \times \boldsymbol{M}).d\mathcal{A}$ and choosing a path of integration outside the magnetized body, we obtain $\int_s \boldsymbol{M}.d\mathcal{A} = 0$, so we can identify $\boldsymbol{J}_m$

$$\boxed{\boldsymbol{J}_m = \nabla \times \boldsymbol{M}}$$

We don't know the details of the magnetization currents, but we can measure the mesoscopic average magnetization and the spontaneous magnetization of a sample.

B and H fields in free space; permeability of free space μ_0

When illustrating Ampère's Law we labelled the magnetic field created by the current, measured in Am^{-1} as ***H***. This is the 'magnetic field strength'

Maxwell's equations have another field, the 'magnetic flux density', labelled ***B***, in the equation $\nabla \cdot \boldsymbol{B} = 0$. It is a different quantity with different units. Whenever ***H*** interacts with matter, to generate a force, an energy or an emf, a constant μ_0, the 'permeability of free space' is involved.

In *free space*, the relation between ***B*** and ***H*** is simple. They are numerically proportional to each other

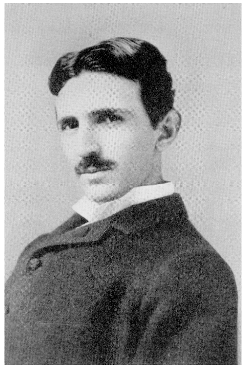
$$\boxed{\boldsymbol{B} = \mu_0 \boldsymbol{H}}$$

B-field
Units Tesla, T

Permeability
of free space
Units TmA^{-1}

H-field
Units $A\ m^{-1}$

In practice you can never mix them up. The differ by almost a million! ($795775 \approx 800000$)

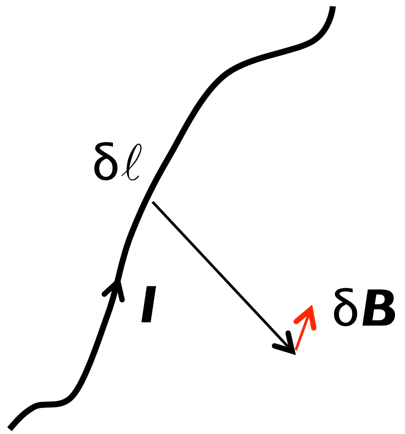


Tesla

μ_0 depends on the definition of the Amp.
It is precisely $4\pi\ 10^{-7}$ T m A^{-1}

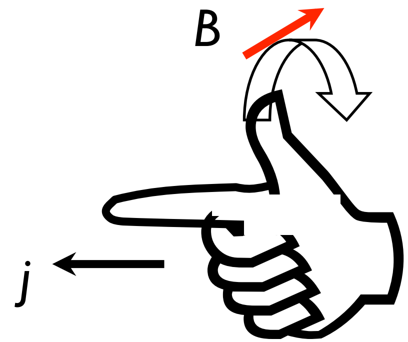
Field due to electric currents

We need a differential form of Ampère's Law;   The Biot-Savart Law



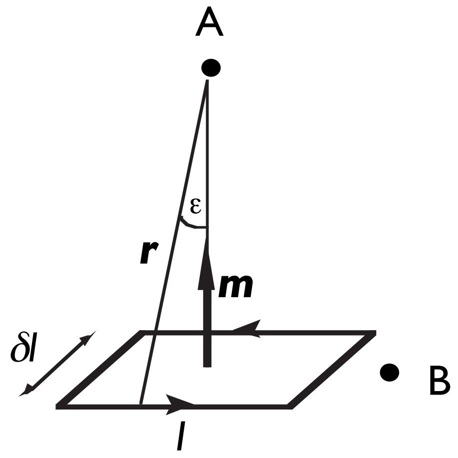
$$\delta \boldsymbol{B} = -\frac{\mu_0}{4\pi} \frac{\boldsymbol{r} \times \boldsymbol{j}}{|r^3|} \delta V$$

$$\delta \boldsymbol{B} = -\frac{\mu_0}{4\pi} I \frac{\boldsymbol{r} \times \delta \boldsymbol{\ell}}{|r^3|}$$



Right-hand corkscrew (for the vector product)

Field due to a small current loop (equivalent to a magnetic moment)



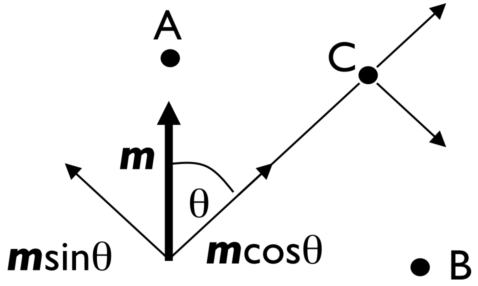
$B_A = 4\delta B \sin\varepsilon$

$$B_B = \frac{\mu_0}{4\pi} I\,\delta l \left\{ \frac{1}{(r - \delta l/2)^2} - \frac{1}{(r + \delta l/2)^2} - \frac{2\sin\epsilon}{r^2} \right\}$$

$\sin\varepsilon = \delta l/2r$ $\quad m = I(\delta l)^2$

$$\boxed{B_A = 2\frac{\mu_0}{4\pi}\frac{\mathbf{m}}{r^3}} \qquad B_A = 2\frac{\mu_0}{4\pi}\frac{\mathbf{m}}{r^3} \qquad \boxed{B_B = \frac{\mu_0}{4\pi}\frac{\mathbf{m}}{r^3}}$$

$$B_B = \frac{\mu_0}{4\pi}\frac{\mathbf{m}}{r^3}$$



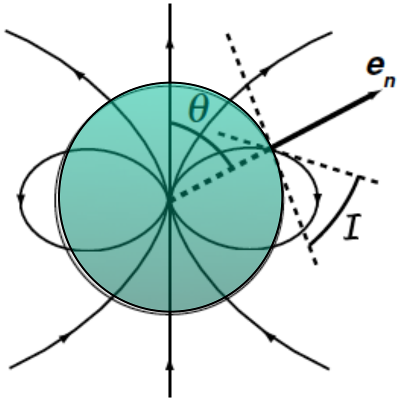
So at a general point C, in spherical coordinates:

$$B_r = 2\left(\frac{\mu_0 \mathbf{m}}{4\pi r^3}\right)\cos\theta; \quad B_\theta = \left(\frac{\mu_0 \mathbf{m}}{4\pi r^3}\right)\sin\theta; \quad B_\phi = 0$$

Field due to a magnetic moment *m*

$$B_r = 2\left(\frac{\mu_0 \mathrm{m}}{4\pi r^3}\right)\cos\theta; \quad B_\theta = \left(\frac{\mu_0 \mathrm{m}}{4\pi r^3}\right)\sin\theta; \quad B_\phi = 0.$$

The Earth's magnetic field is roughly that of a geocentric dipole

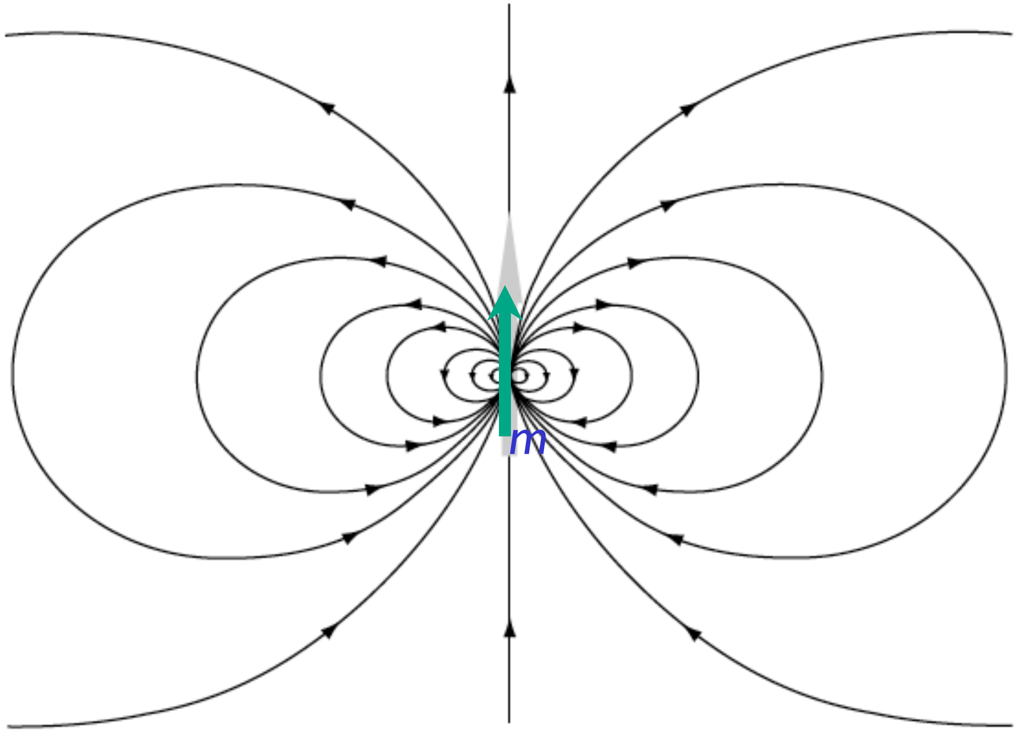


The angle of dip.

$\tan I = B_r / B_\theta = 2\cot\theta$

$dr/rd\theta = 2\cot\theta$

Solutions are
$r = c\,\sin^2\theta$

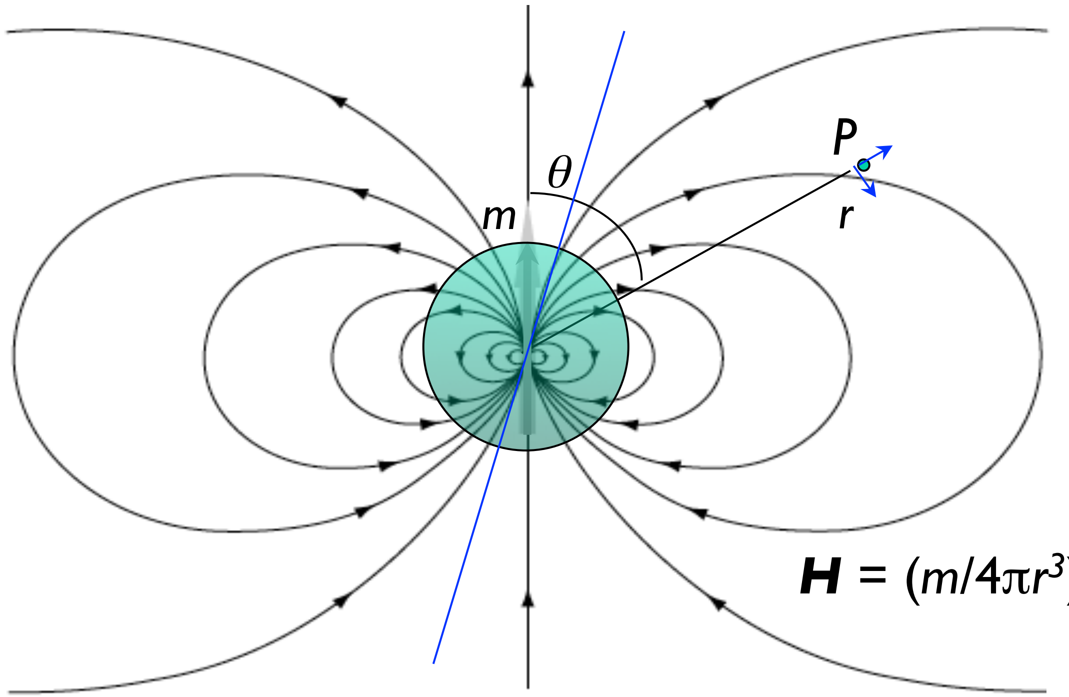


Equivalent forms

$$\boldsymbol{B} = \frac{\mu_0 \mathrm{m}}{4\pi r^5}\left[3xz\mathbf{e}_x + 3yz\mathbf{e}_y + (3z^2 - r^2)\mathbf{e}_z\right]$$

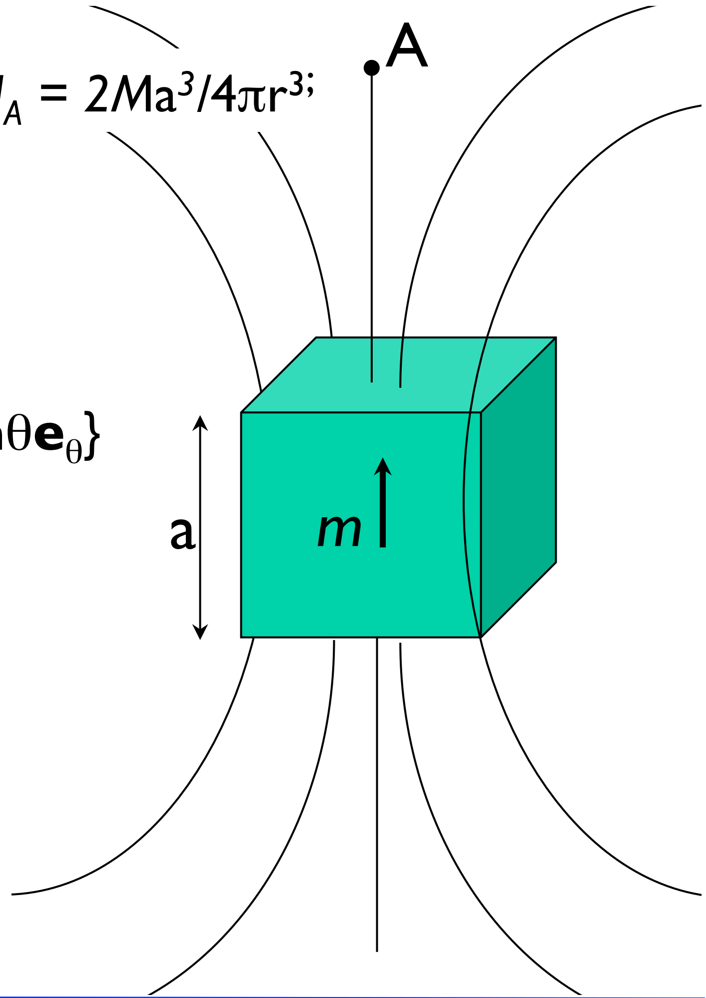
$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\left[3\frac{(\mathrm{m}\cdot\boldsymbol{r})\boldsymbol{r}}{r^5} - \frac{\mathrm{m}}{r^3}\right]$$

Magnetic field due to a moment *m*; Scaling



Why does magnetism lend itself to miniaturization ?

$H_A = 2Ma^3/4\pi r^3$;



$\boldsymbol{H} = (m/4\pi r^3)\{2\cos\theta\,\mathbf{e}_r - \sin\theta\,\mathbf{e}_\theta\}$

Just like the field of an electric dipole

If a = 0.01m, r = 2a, M = 1 MA m^{-1}

$H_A = 2M/16\pi =$ 40 kA m^{-1}

What is the average magnetization of the Earth ?

Magnet-generated fields just depend on M. They are *scale-independent*

3. Magnetic Fields

Magnetic flux density - *B*

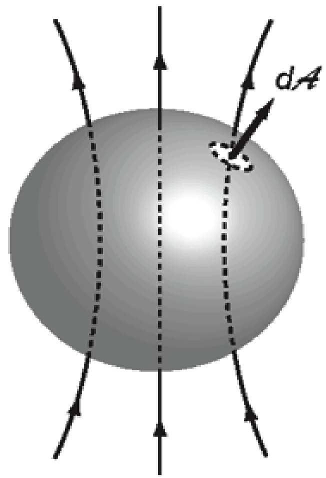
Now we discuss the fundamental field in magnetism.

Magnetic poles, analogous to electric charges, *do not exist*. This truth is expressed in Maxwell's equation

$$\boxed{\nabla . \boldsymbol{B} = 0.}$$

This means that the lines of the *B*-field always form complete loops; they never start or finish on magnetic charges, the way the electric *E*-field lines can start and finish on +ve and -ve electric charges.

The same can be written in integral form over *any closed surface S*



$$\boxed{\int_S \boldsymbol{B} \bullet d\mathcal{A} = 0} \qquad \text{(Gauss's law).}$$

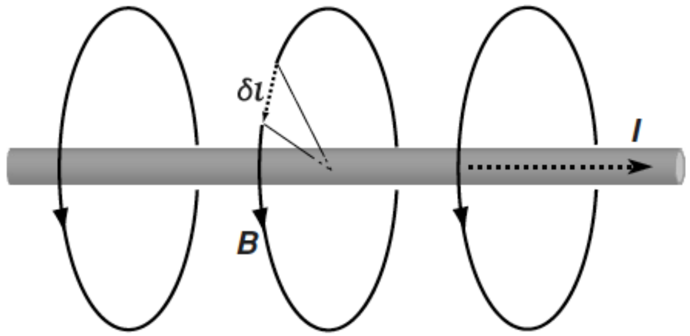
The flux of ***B*** across a surface is $\Phi = \int \boldsymbol{B} \bullet d\mathcal{A}$. Units are Webers (Wb).
The net flux across any closed surface is zero.
B is known as the flux density; units are Teslas. (T = Wb m^{-2})

Flux quantum Φ_0 = 2.07 10^{15} Wb (Tiny)

The B-field

*Sources of **B***

- electric currents in conductors
- moving charges
- magnetic moments
- time-varying electric fields. (Not in *magnetostatics*)



$B = \mu_0 I/2\pi r$

In a steady state: Maxwell's equation

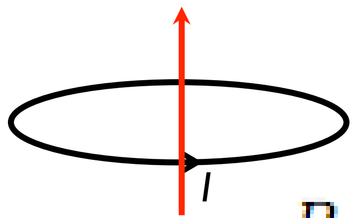
$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j}$$

$$\begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ B_x & B_Y & B_Z \end{vmatrix}$$

$$\oint \boldsymbol{B} \cdot \mathrm{d}l = \mu_0 I$$

(*Stokes theorem*;

$$\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{e}_n dr^2 = \oint \mathbf{A} \cdot d\ell$$



$$B_0 = \frac{\mu_0 I}{2a}$$

Field at center of current loop

Forces between conductors; Definition of the Amp

$\boxed{\boldsymbol{F} = q(\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B})}$

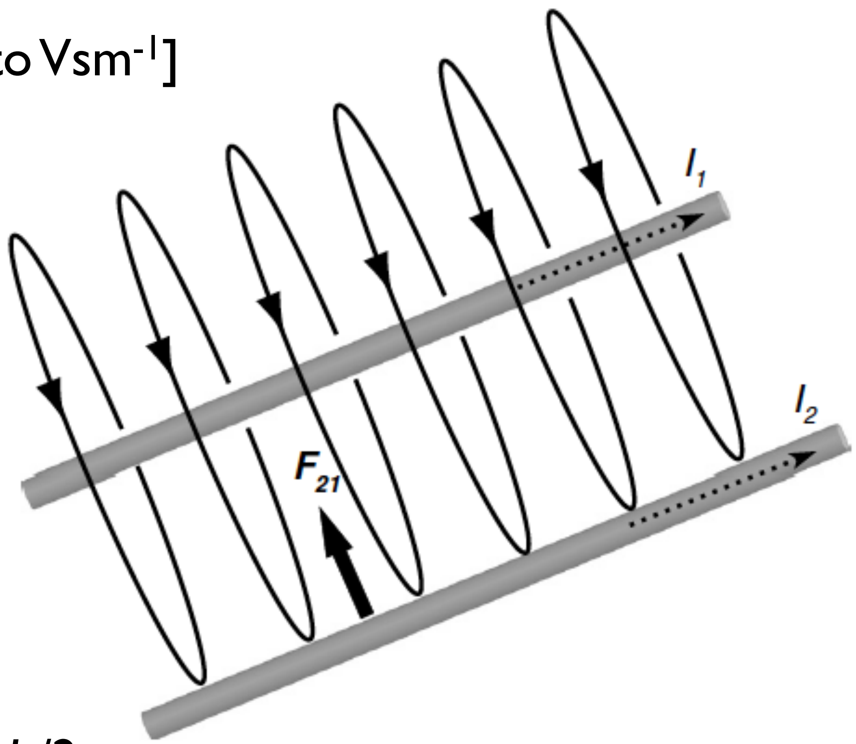
Lorentz expression. Note the Lorentz force does no work on the charge

Gives dimensions of ***B*** and ***E***. [T is equivalent to Vsm^{-1}]

If $\boldsymbol{E} = 0$ the force on a straight wire arrying a current ***I*** in a uniform field ***B*** is $F = BI\ell$

The force between two parallel wires each carrying one ampere is precisely $2\ 10^{-7}$ N m^{-1}. (Definition of the Amp)

The field at a distance 1 m from a wire carrying a current of 1 A is 0.2 μT

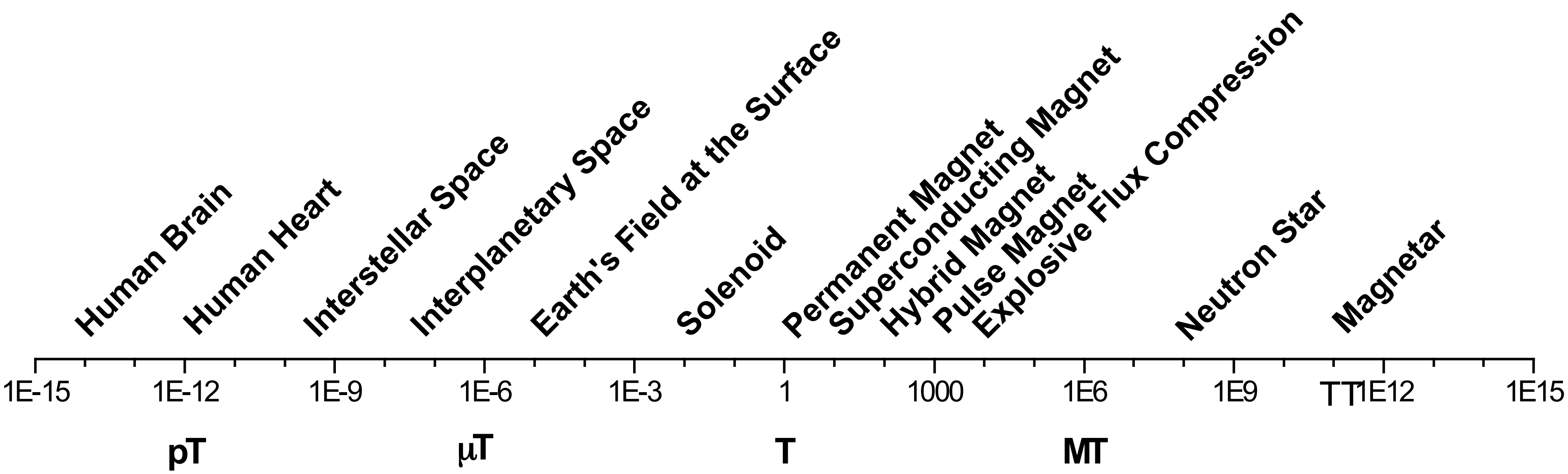


$B = \mu_0 I_1/2\pi r$

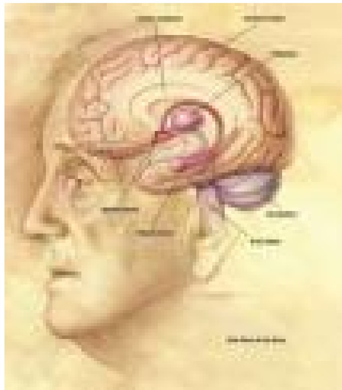
Force per meter = $\mu_0 I_1 I_2/2\pi r$

The range of magnitude of *B*

- The tesla is a very large unit
- Largest continuous field acheived in a lab was 45 T

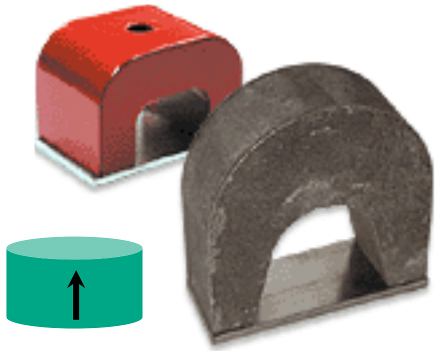


Typical values of *B*



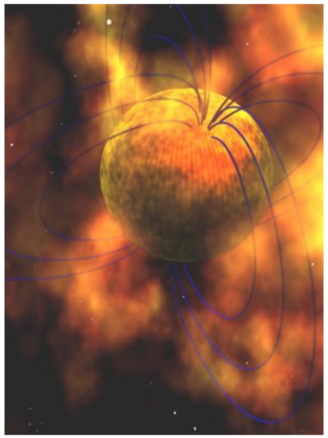
Human brain 1 fT

Earth 50 μT



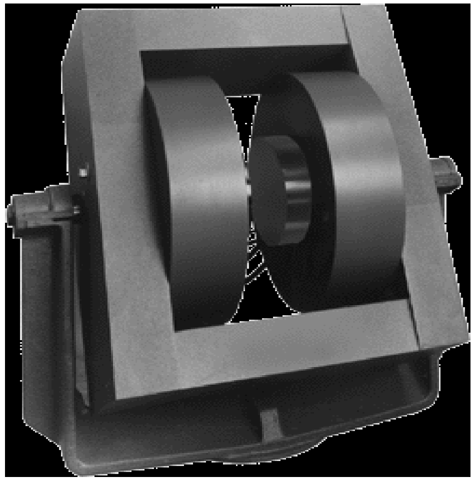
Permanent magnets 0.5 T

Superconducting magnet 10 T



Magnetar 10^{12} T

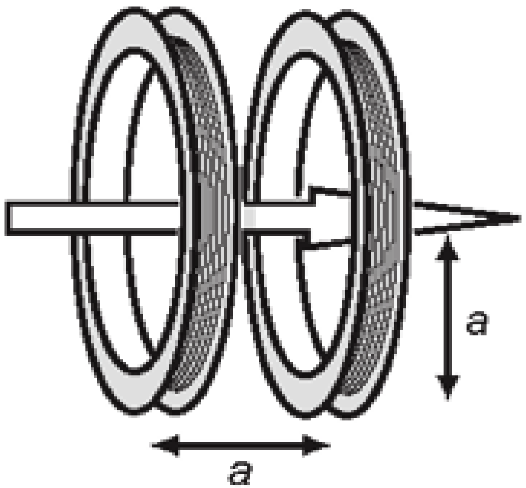
Helmholtz coils 0.01 T



Electromagnet 1 T

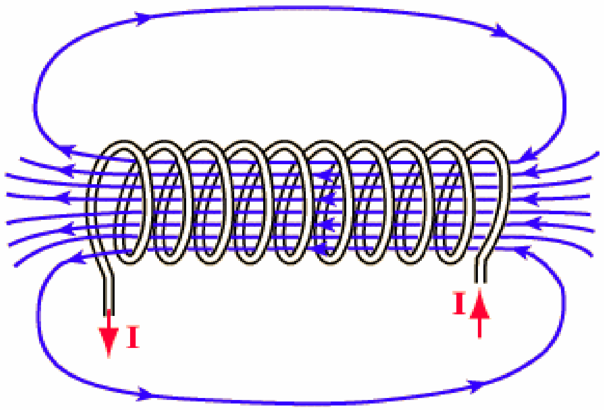
Sources of uniform magnetic fields

Helmholtz coils



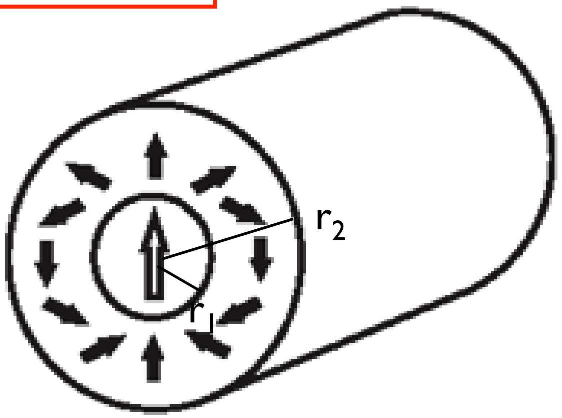
$$B = (4/5)^{3/2}\mu_0 N I/a$$

Long solenoid



$$B = \mu_0 n I$$

Halbach cylinder



$$B_r = \left(\frac{\mu_0 \lambda}{2\pi r^2}\right)\cos\theta, \quad B_\theta = \left(\frac{\mu_0 \lambda}{2\pi r^2}\right)\sin\theta, \quad B_z = 0.$$

$$B = \mu_0 M \ln(r_2/r_1)$$

The *H*-field.

Ampère's law for the field is free space is $\nabla \times \boldsymbol{B} = \mu_0(\boldsymbol{j}_c + \boldsymbol{j}_m)$ but $\boldsymbol{j}_m$ cannot be measured !

Only the conduction current $\boldsymbol{j}_c$ is accessible.

We showed on slide 16 that $\boldsymbol{J}_m = \nabla \times \boldsymbol{M}$

Hence $\nabla \times (\boldsymbol{B}/\mu_0 - \boldsymbol{M}) = \mu_0\, \boldsymbol{j}_c$

We can retain Ampère's law is a usable form provided we define $\boldsymbol{H} = \boldsymbol{B}/\mu_0 - \boldsymbol{M}$

Then $$\boxed{\nabla \times \boldsymbol{H} = \mu_0\, \boldsymbol{j}_c}$$

And $$\boxed{\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})}$$

The *H*-field.

The *H*-field plays a critical role in condensed matter.
The state of a solid reflects the local value of ***H***.
Hysteresis loops are plotted as $M(H)$

Unlike ***B***, ***H*** is not solenoidal. It has sources and sinks in a magnetic material wherever the magnetization is nonuniform.

$$\nabla . \boldsymbol{H} = - \nabla . \boldsymbol{M}$$

The sources of ***H*** (magnetic charge, q_{mag}) are distributed

— in the bulk with charge density $-\nabla . \boldsymbol{M}$
— at the surface with surface charge density $\boldsymbol{M} . \boldsymbol{e}_n$

*Coulomb approach to calculate **H*** (in the absence of currents)

Imagine ***H*** due to a distribution of magnetic charges q_m (units Am) so that

$$\boldsymbol{H} = q_m \boldsymbol{r}/4\pi r^3 \quad \text{[just like electrostatics]}$$

Potentials for ***B*** and ***H***

It is convenient to derive a field from a potential, by taking a spatial derivative. For example $\boldsymbol{E} = -\nabla\varphi_e(\boldsymbol{r})$ where $\varphi_e(\boldsymbol{r})$ is the electric potential. Any constant φ_0 can be added.

For ***B***, we know from Maxwell's equations that $\nabla.\boldsymbol{B} = 0$. There is a vector identity $\nabla\boldsymbol{\cdot}\,\nabla\times\boldsymbol{X} \equiv 0$. Hence, we can derive $\boldsymbol{B}(\boldsymbol{r})$ from a *vector potential* $\boldsymbol{A}(\boldsymbol{r})$ (units Tm),

$$\boxed{\boldsymbol{B}(\boldsymbol{r}) = \nabla \times \boldsymbol{A}(\boldsymbol{r})}$$

The gradient of any scalar *f* can be added to ***A*** (a *gauge* transformation) This is because of another vector identity $\nabla\times\nabla\boldsymbol{\cdot}f \equiv 0$.

Generally, $\boldsymbol{H}(\boldsymbol{r})$ *cannot* be derived from a potential. It satisfies Maxwell's equation $\nabla \times \boldsymbol{H} = \boldsymbol{j}_c + \partial\mathbf{D}/\partial t$. In a static situation, when there are *no conduction currents* present, $\nabla \times \boldsymbol{H} = 0$, and

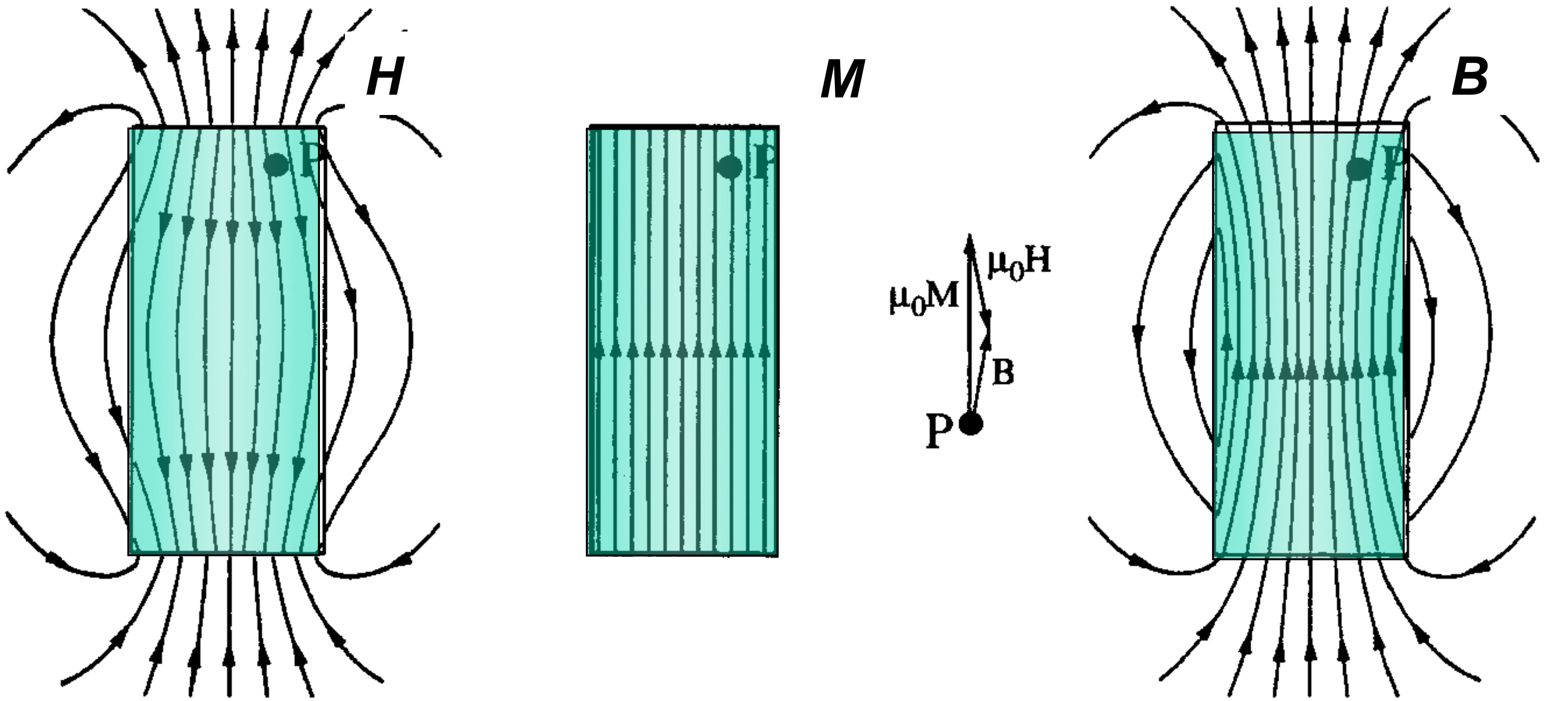
$$\boxed{\boldsymbol{H}(\boldsymbol{r}) = -\nabla\varphi_m(\boldsymbol{r})}$$

In these special conditions, it is possible to derive $\boldsymbol{H}(\boldsymbol{r})$ from a magnetic scalar potential φ_m (units A). We can imagine that ***H*** is derived from a distribution of magnetic 'charges' $\pm q_m$.

Relation between ***B*** and ***H*** in a material

The general relation between ***B, H*** and ***M*** is

$$\boxed{\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})} \qquad \text{i.e. } \boldsymbol{H} = \boldsymbol{B}/\mu_0 - \boldsymbol{M}$$

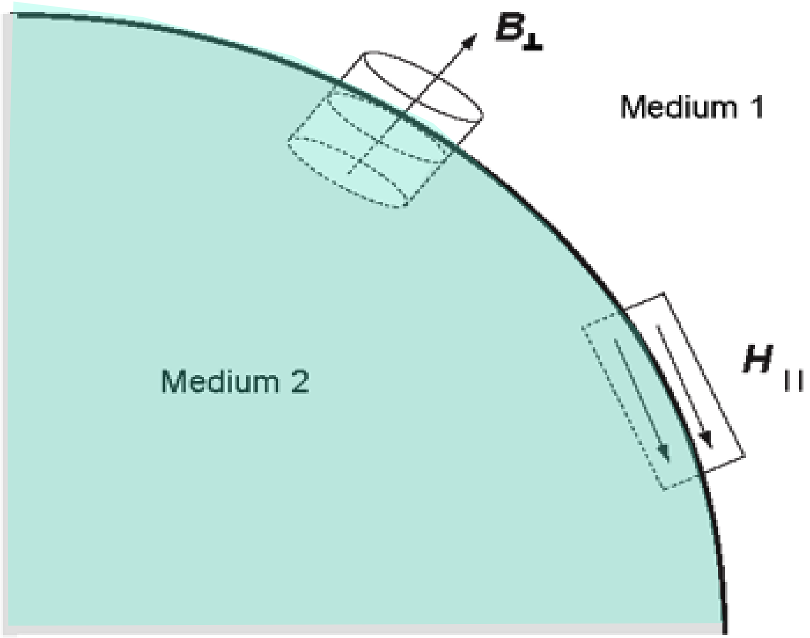


We call the *H*-field due to a magnet; — *stray field* outside the magnet

— *demagnetizing field*, H_d, inside the magnet

Boundary conditions

Conditions on the fields



It follows from Gauss' s law

$$\int_S \boldsymbol{B} \cdot d\mathcal{A} = 0$$

that the *perpendicular component* of ***B*** is continuous. $(\boldsymbol{B}_1 - \boldsymbol{B}_2) \cdot \boldsymbol{e}_n = 0$

It follows from from Ampère' s law

$$\oint \boldsymbol{H} . d\boldsymbol{l} = I = 0$$

that the *parallel component* of ***H*** is continuous. since there are no conduction currents on the surface. $(\boldsymbol{H}_1 - \boldsymbol{H}_2) \times \mathbf{e}_n = 0$

Conditions on the potentials

Since $\int_S \boldsymbol{B} \cdot d\mathcal{A} = \oint \boldsymbol{A} \cdot d\boldsymbol{l}$ (Stoke' s theorem)

$$(\boldsymbol{A}_1 - \boldsymbol{A}_2) \times \mathbf{e}_n = 0$$

The scalar potential is continuous $\varphi_{m1} = \varphi_{m2}$

Boundary conditions in linear, isotropic homogeneous media

In LIH media, $\boldsymbol{B} = \mu_0\mu_r\boldsymbol{H}$

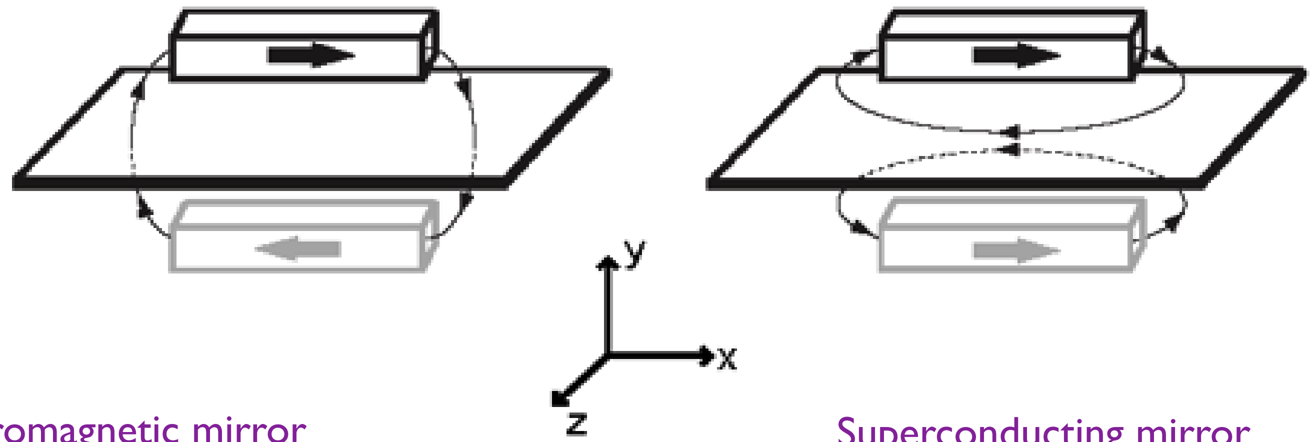
Hence $\boldsymbol{B}_1\boldsymbol{e}_n = \boldsymbol{B}_2\boldsymbol{e}_n$

$$\boldsymbol{H}_1\boldsymbol{e}_n = \mu_{r2}/\mu_{r1}\boldsymbol{H}_2\boldsymbol{e}_n$$

So field lies ~ perpendicular to the surface of soft iron but parallel to the surface of a superconductor.

Diamagnets produce weakly repulsive images.

Paramagnets produce weakly attractive images.

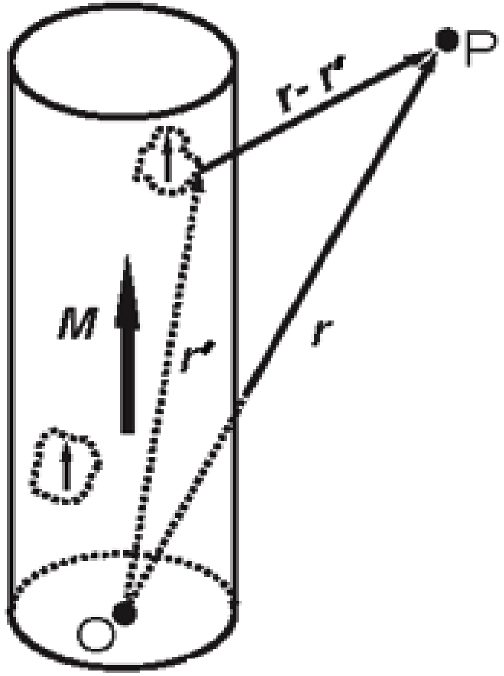


Soft ferromagnetic mirror

Superconducting mirror

Field calculations – Volume integration

*Integrate over volume distribution of **M***



Sum over fields produced by each magnetic dipole element ***M***d^3*r*.

Using

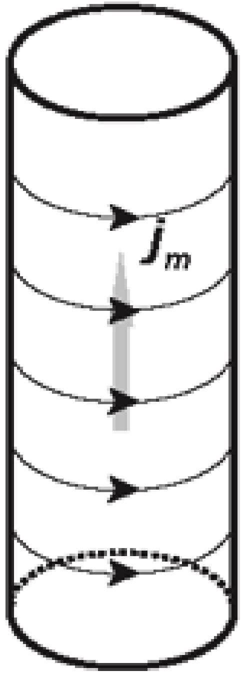
$$\boldsymbol{B} = \frac{\mu_0}{4\pi}\left[3\frac{(\mathfrak{m}.\boldsymbol{r})\boldsymbol{r}}{r^5} - \frac{\mathfrak{m}}{r^3}\right]$$

Gives

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\left[\int\left\{\frac{3\boldsymbol{M}(\boldsymbol{r}').(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^5}(\boldsymbol{r}-\boldsymbol{r}') - \frac{\boldsymbol{M}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} + \frac{2}{3}\mu_0\boldsymbol{M}(\boldsymbol{r}')\delta(\boldsymbol{r}-\boldsymbol{r}')\right\}d^3r'\right]$$

(Last term takes care of divergences at the origin)

Field calculations – Ampèrian approach



Consider bulk and surface current distributions

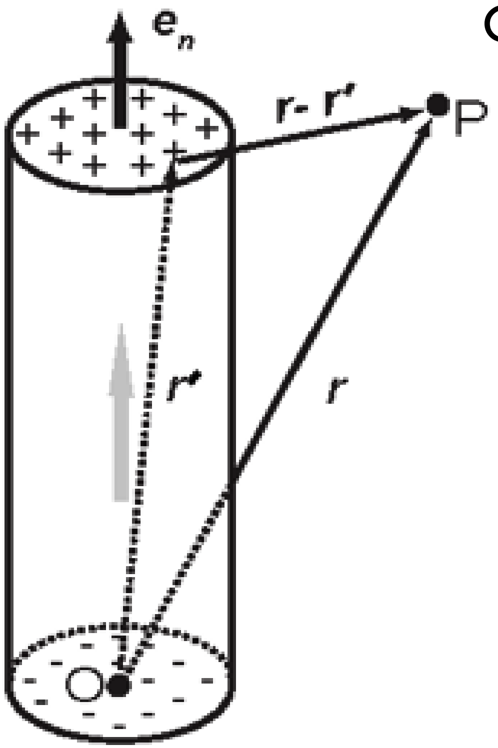
$\boldsymbol{j}_m = \nabla \times \boldsymbol{M}$ *and* $\boldsymbol{j}_{ms} = \boldsymbol{M} \times \mathbf{e}_n$

Biot-Savart law gives

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\left\{\int \frac{(\nabla \times \boldsymbol{M}) \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} d^3r' + \int \frac{(\boldsymbol{M} \times \mathbf{e}_n) \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} d^2r'\right\}$$

For uniform $\boldsymbol{M}$, the Bulk term is zero since $\nabla \times \boldsymbol{M} = 0$

Field calculations – Coulombian approach



Consider bulk and surface magnetic charge distributions

$$\rho_m = -\nabla.\boldsymbol{M} \qquad \textit{and} \qquad \rho_{ms} = \boldsymbol{M}.\boldsymbol{e_n}$$

H field of a small charged volume element V is

$$\delta\boldsymbol{H} = (\rho_m \boldsymbol{r}/4\pi r^3)\ \delta V$$

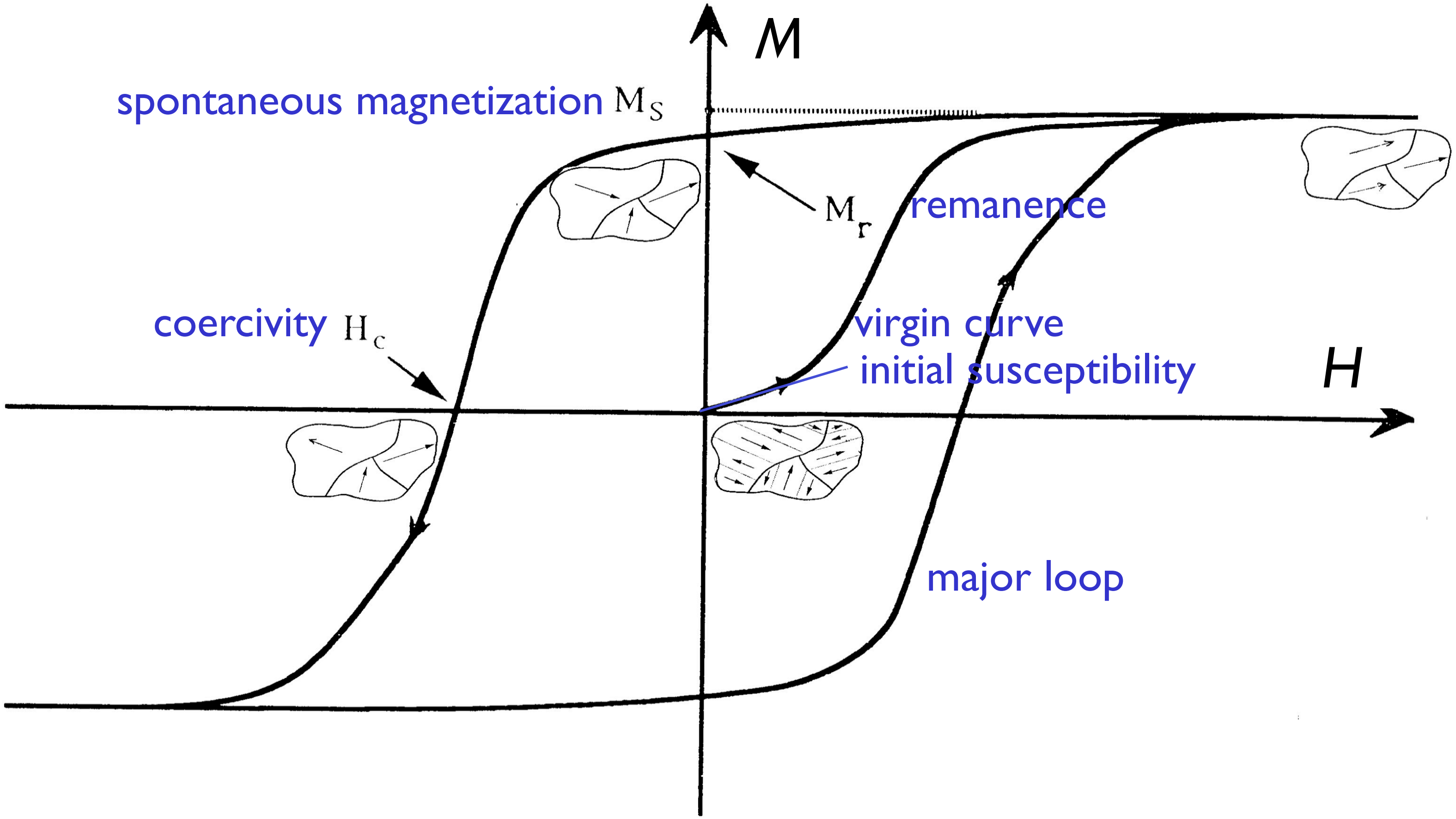
So

$$\boldsymbol{H}(\boldsymbol{r}) = \frac{1}{4\pi}\left\{-\int_V \frac{(\nabla.\boldsymbol{M})(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}d^3r' + \int_S \frac{\boldsymbol{M}.\mathbf{e}_n(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}d^2r'\right\}$$

For a uniform magnetic distribution *the first term is zero.*

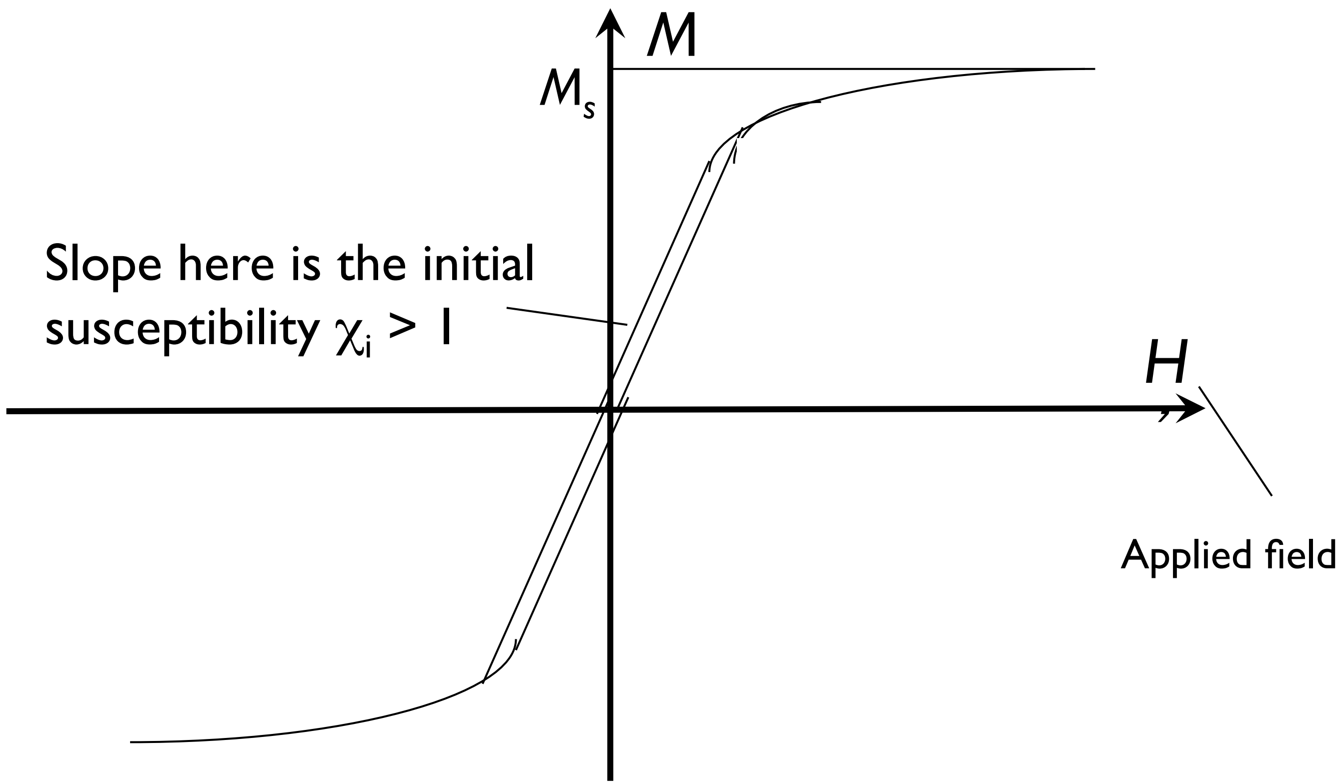
$$\nabla.\boldsymbol{M} = 0$$

Hysteresis loop; permanent magnet (hard ferromagnet)



A broad $M(H)$ loop

Hysteresis loop; temporary magnet (soft ferromagnet)



Susceptibility is defined as $\chi = M/H$

A narrow $M(H)$ loop

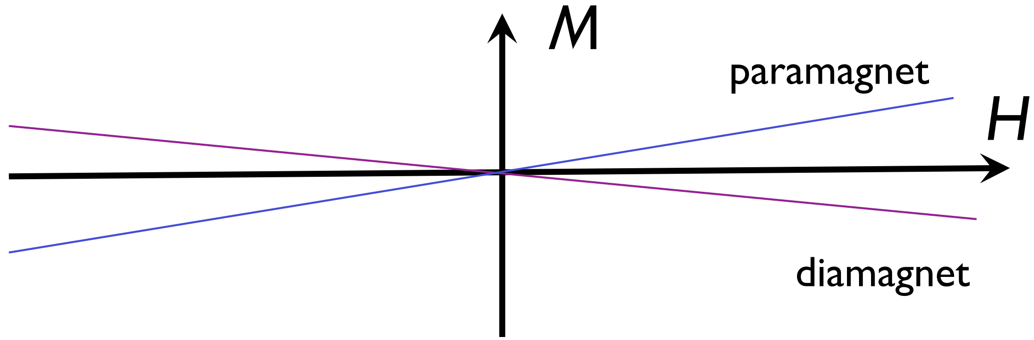
Paramagnets and diamagnets; antiferromagnets.

Only a few elements and alloys are ferromagnetic. (See the magnetic periodic table). The atomic moments in a ferromagnet order spontaneosly parallel to eachother.

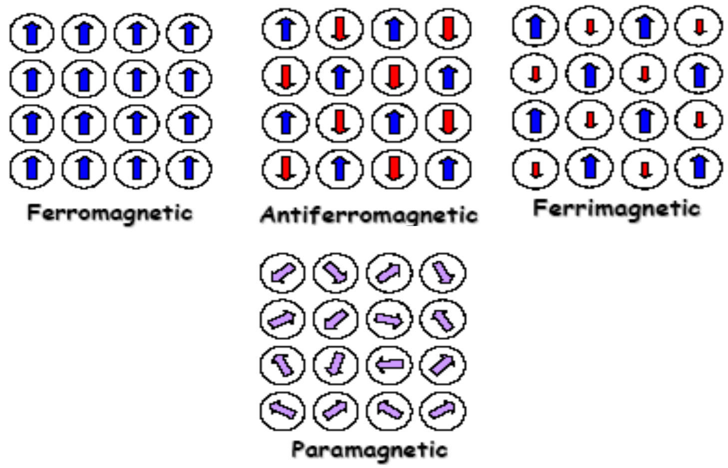
Most have no spontaneous magnetization, and they show only a very weak response to a magnetic field.

Here $|\chi| << 1$

χ is 10^{-4} - 10^{-6}



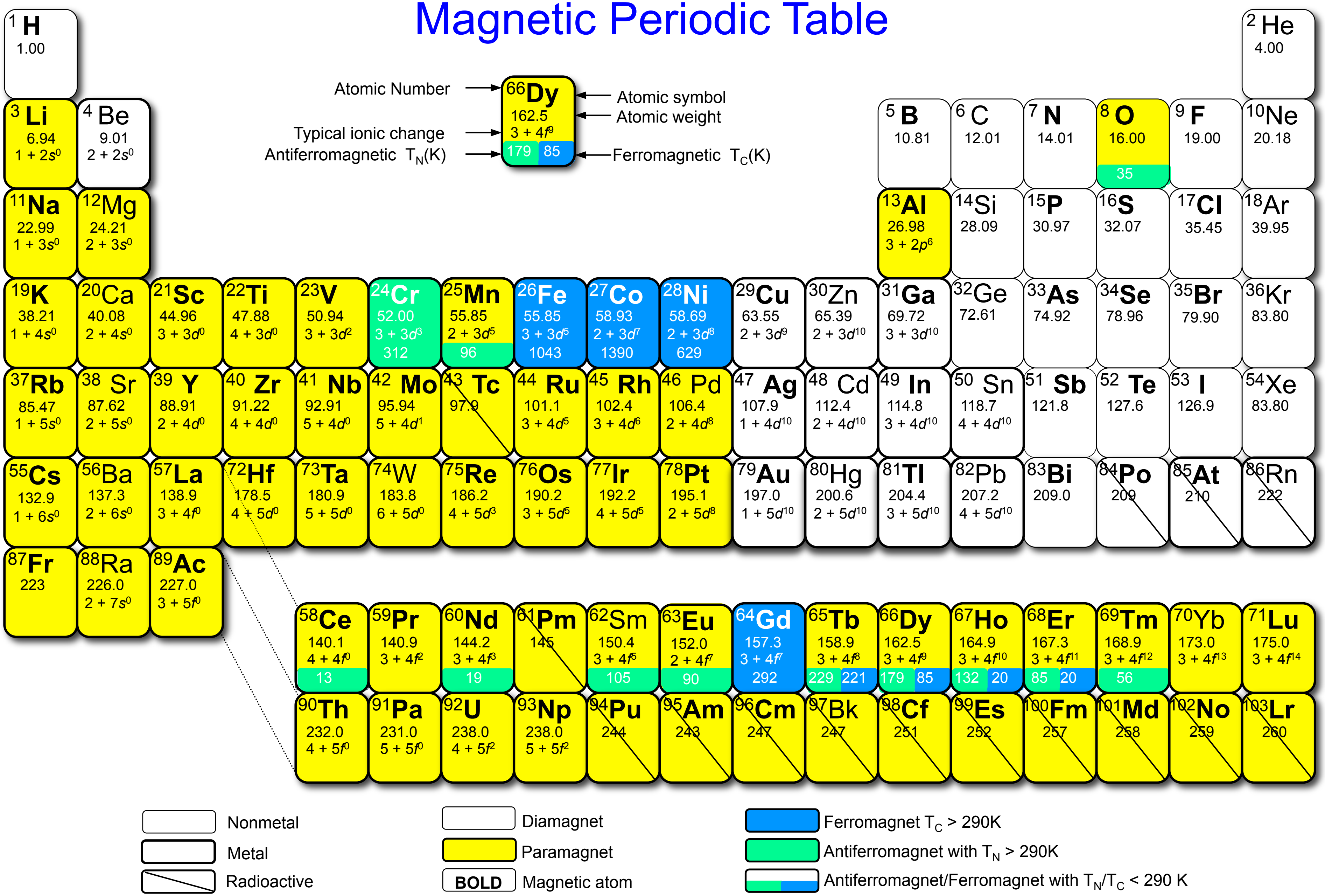
A few elements and oxides are antiferromagnetic. The atomic moments order spontaneously antiparallel to eachother.



Ordered $T < T_c$

Disordered $T > T_c$

Magnetic Periodic Table



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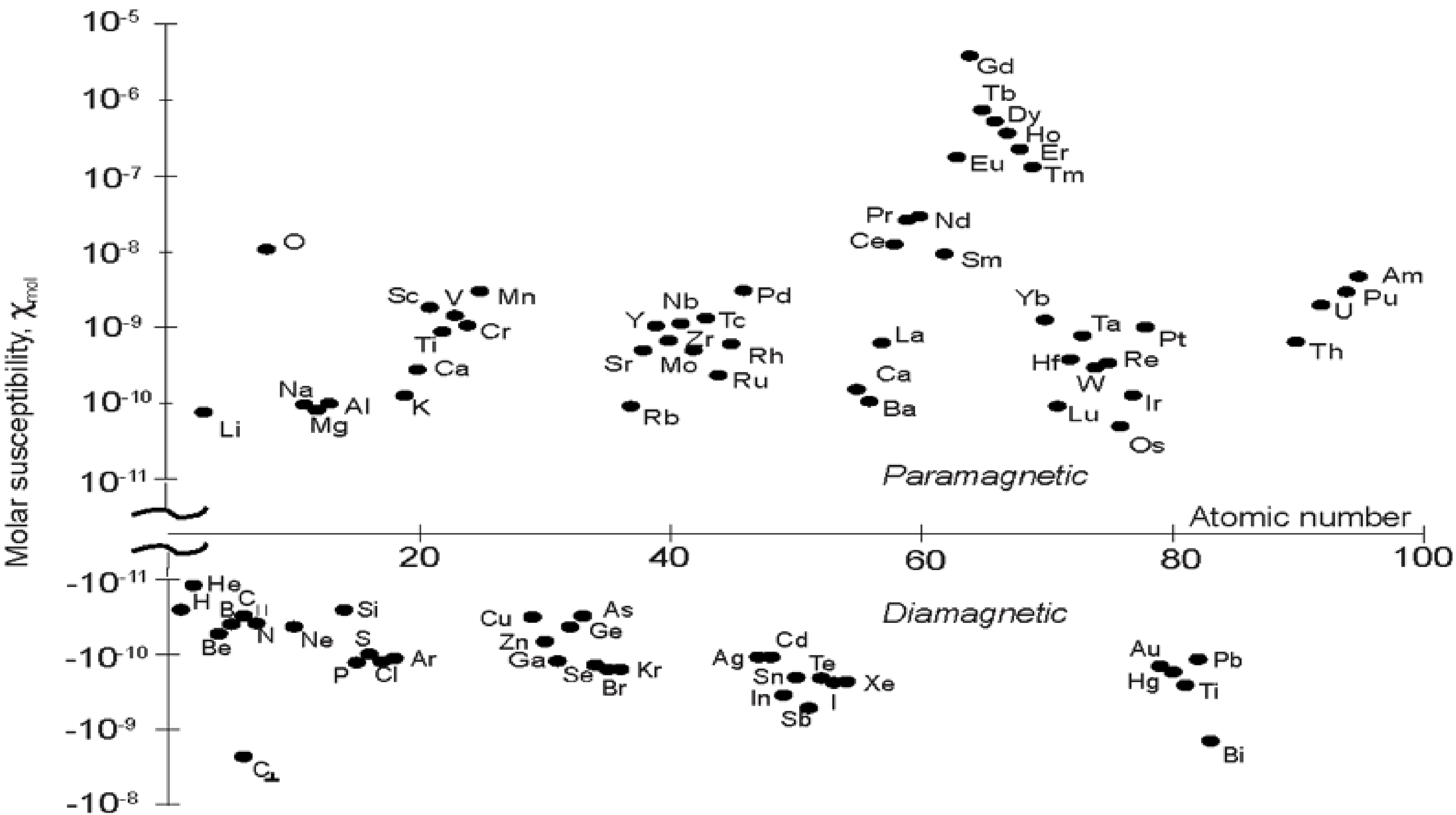
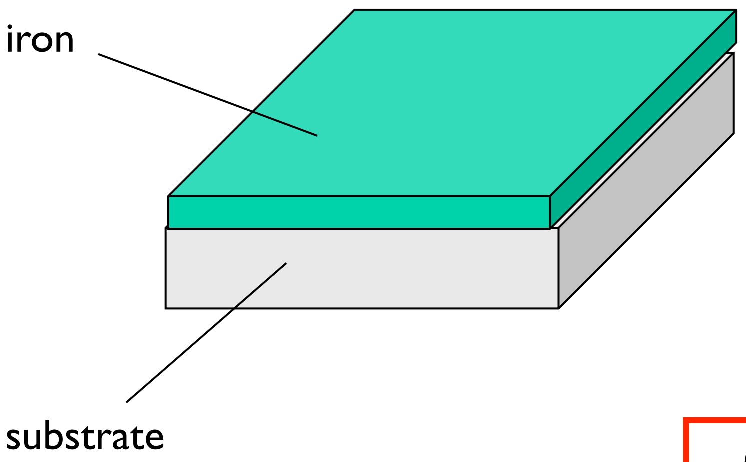
Susceptibilities of the elements

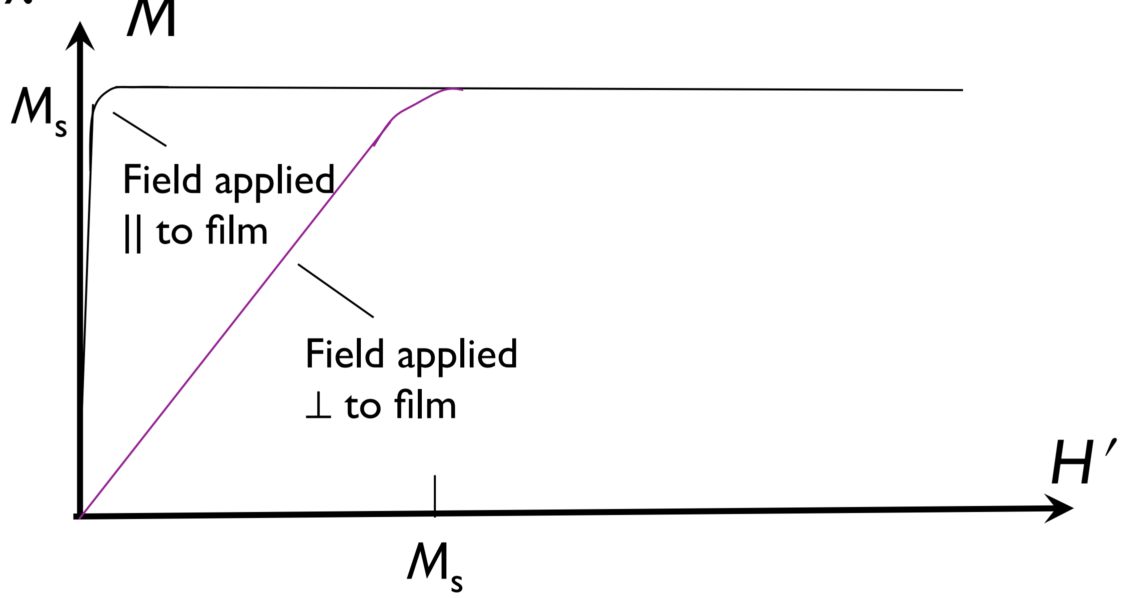


Magnetic fields - Internal and applied fields

In ALL these materials, the *H*-field acting inside the material is not the one you apply. These are *not* the same. If they were, any applied field would instantly saturate the magnetization of a ferromagnet when $\chi > 1$.

Consider a thin film of iron.





$$\boxed{\boldsymbol{H} = \boldsymbol{H}' + \boldsymbol{H}_d}$$

Internal field

External field

Demagnetizing field

$M_s = 1.72\ \mathrm{MAm^{-1}}$

Demagnetizing field in a material - $\boldsymbol{H}_d$

The demagnetizing field depends on the *shape* of the sample and the direction of magnetization.

For simple uniformly-magnetized shapes (ellipsoids of revolution) the demagnetizing field is related to the magnetization by a proportionality factor $\mathcal{N}$ known as the *demagnetizing factor*. The value of $\mathcal{N}$ can never exceed 1, nor can it be less than 0.

$$\boxed{\boldsymbol{H}_d = - \mathcal{N}\boldsymbol{M}}$$

More generally, this is a tensor relation. $\mathcal{N}$ is then a 3 x 3 matrix, with trace 1. That is

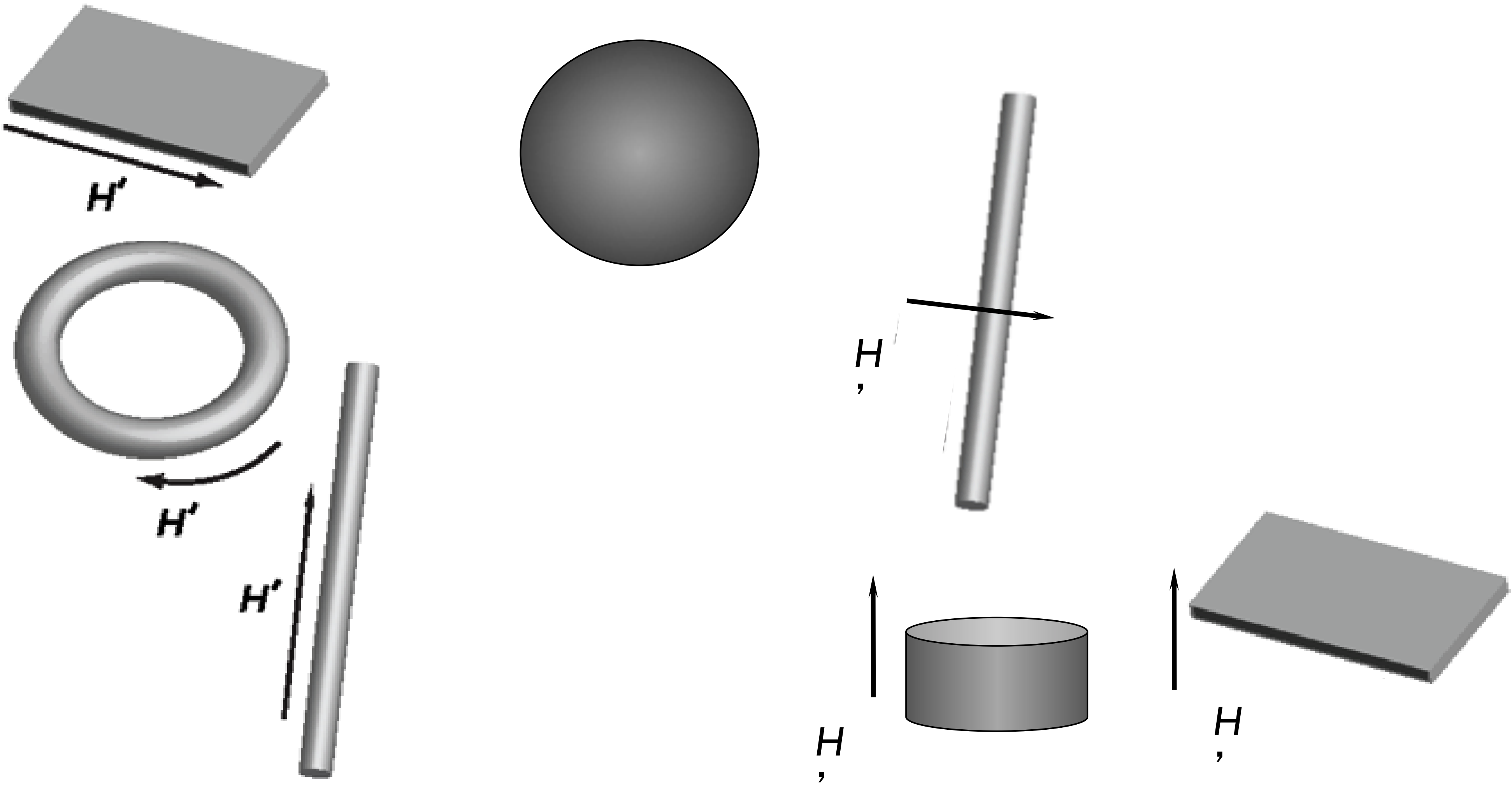
$$\boxed{\mathcal{N}_x + \mathcal{N}_y + \mathcal{N}_z = 1}$$

Note that the internal field *H* is always less than the applied field *H'* since

$$\boxed{\boldsymbol{H} = \boldsymbol{H}' - \mathcal{N}\boldsymbol{M}}$$

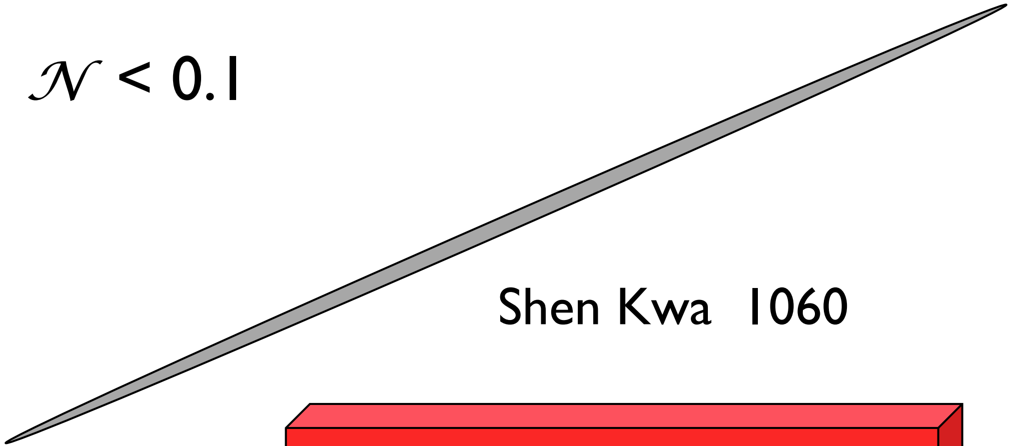
Demagnetizing factor $\mathcal{N}$ for special shapes.

Shapes $\mathcal{N} = 0$ | $\mathcal{N} = 1/3$ | $\mathcal{N} = 1/2$ | $\mathcal{N} = 1$



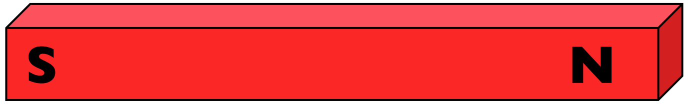
The shape barrier

$\mathcal{N} < 0.1$



Daniel Bernouilli
1743

Shen Kwa 1060

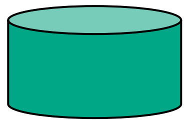


Gowind Knight 1760

T. Mishima
1931

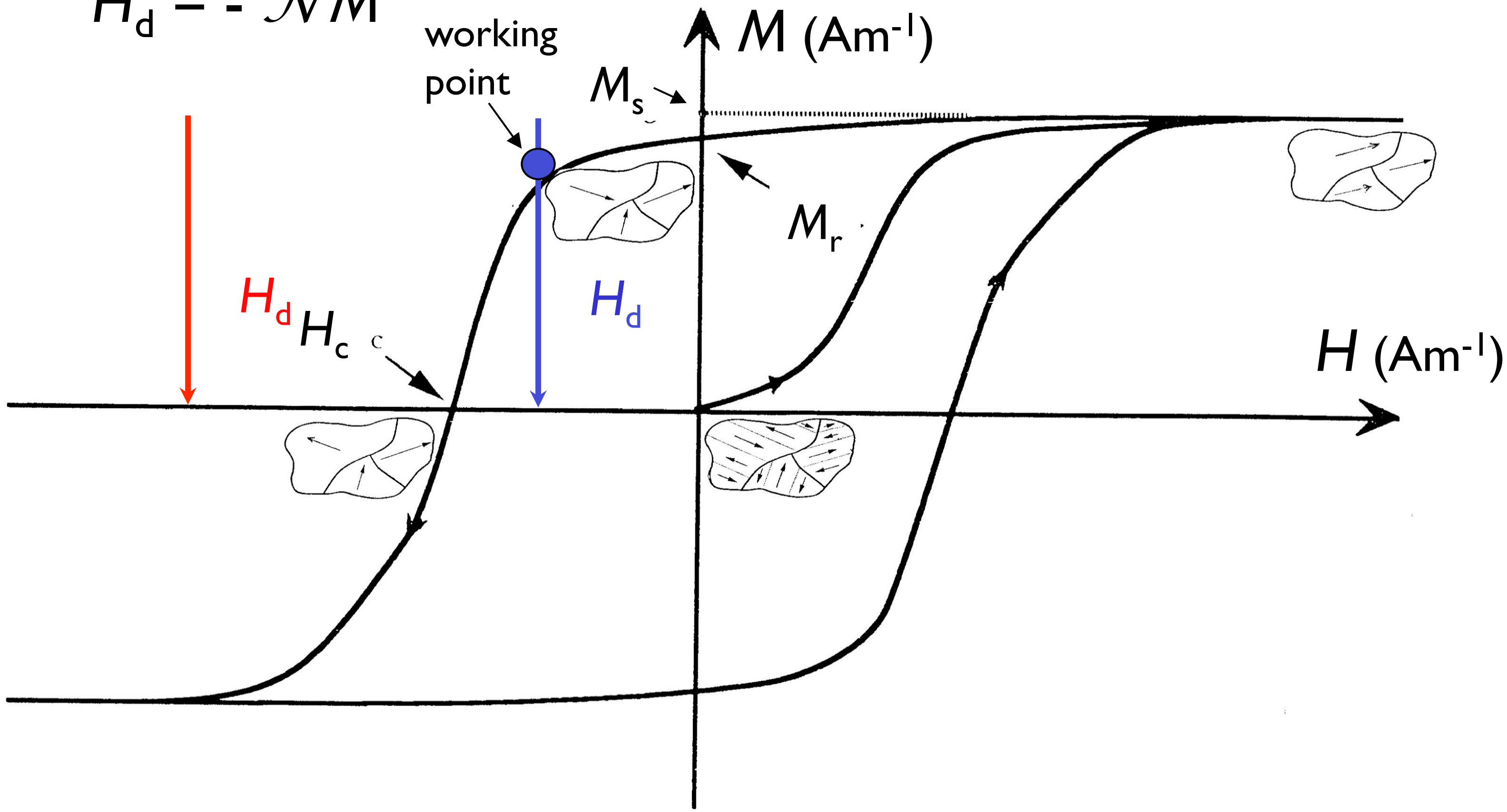
$\mathcal{N} = 0.5$

New icon for permanent magnets! $\Rightarrow$

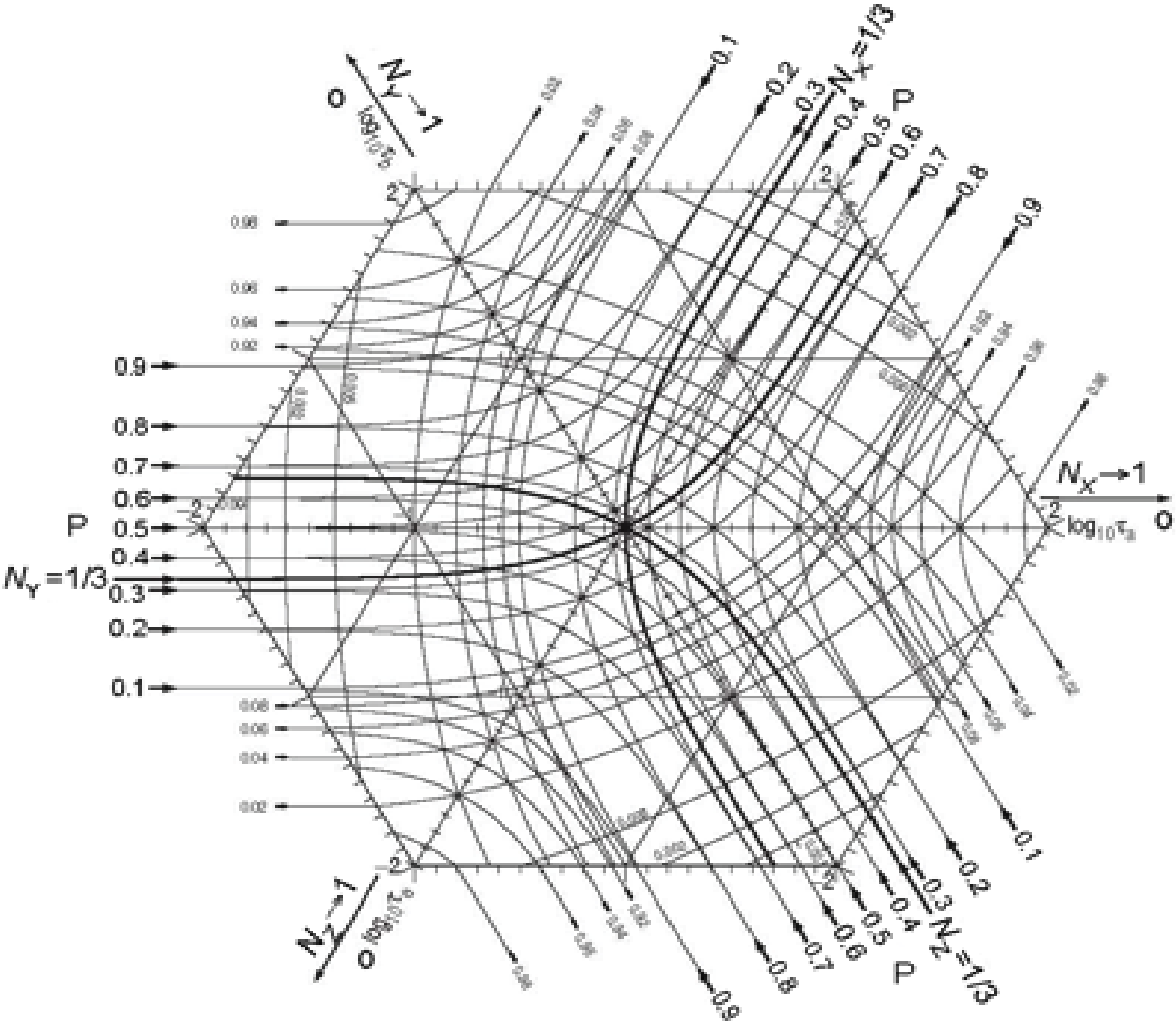


The shape barrier ovecome !

$H_d = - \mathcal{N} M$



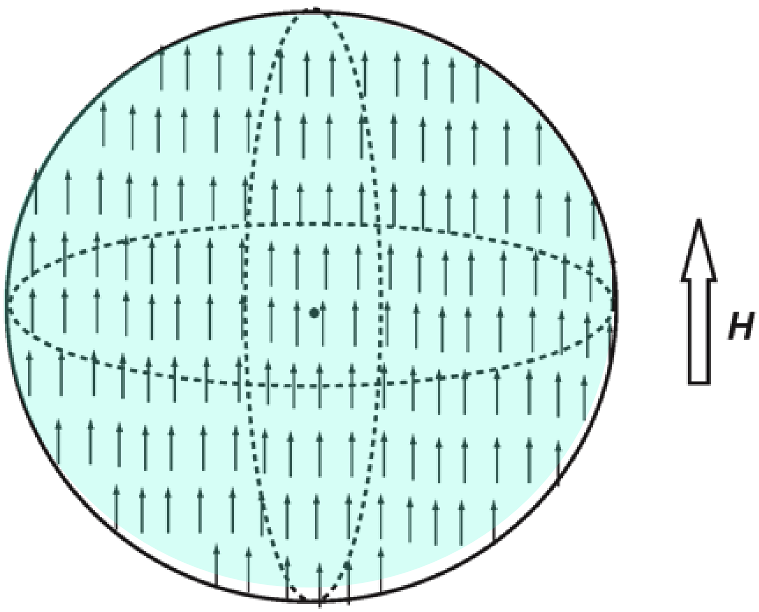
Demagnetizing factors $\mathcal{N}_i$ for a general ellipsoid.

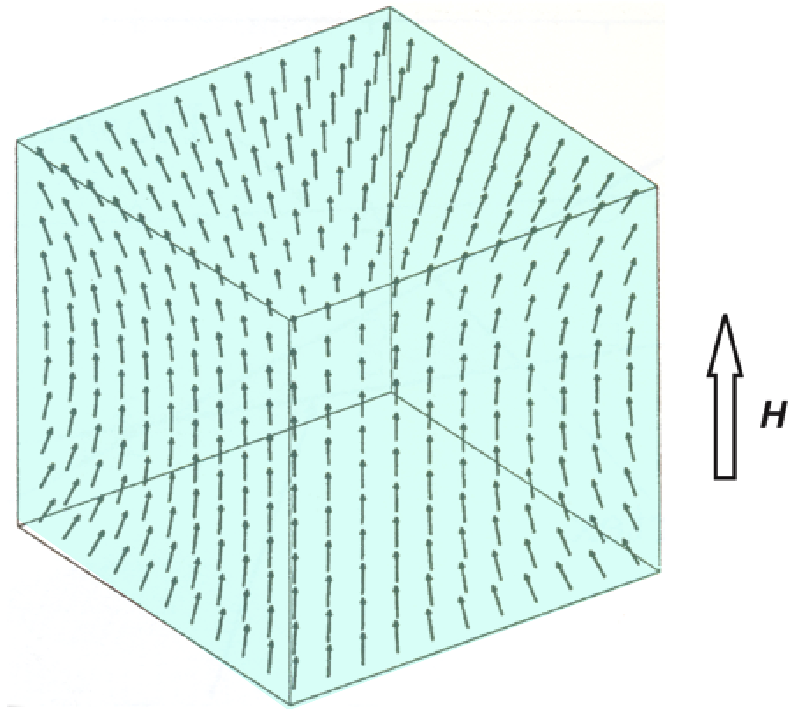


Calculate from analytial formulae

$$\mathcal{N}_x + \mathcal{N}_y + \mathcal{N}_z = 1$$

Other shapes cannot be unoformly magnetized..





When measuring the magnetization of a sample ***H*** is always taken as the independent variable, ***M*** = ***M***(***H***).

Paramagnetic and ferromagnetic responses

Susceptibility of linear, isotropic and homogeneous (LIH) materials

$\boldsymbol{M} = \chi'\boldsymbol{H}'$ χ' is external susceptibility (no units)

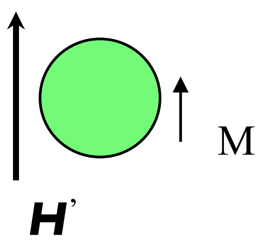
It follows that from $\boldsymbol{H} = \boldsymbol{H}' + \boldsymbol{H}_d$ that

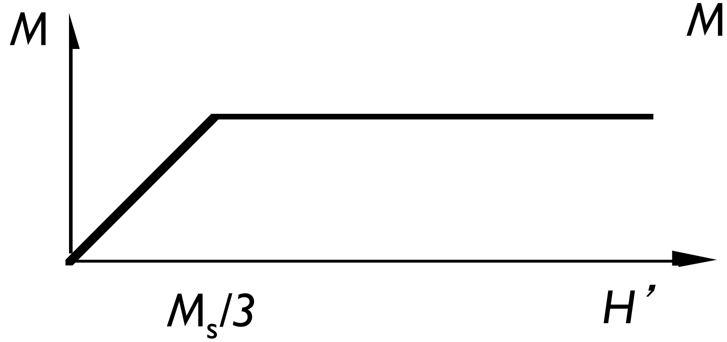
$$\boxed{1/\chi = 1/\chi' - \mathcal{N}}$$

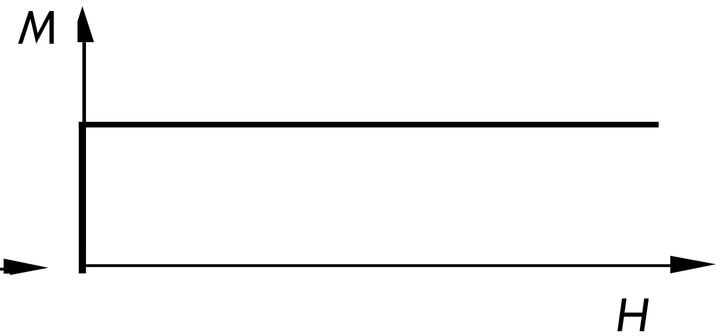
Typical paramagnets and diamagnets: $\chi \approx \chi'$ (10^{-5} to 10^{-3})

Paramagnets close to the Curie point and ferromagnets:

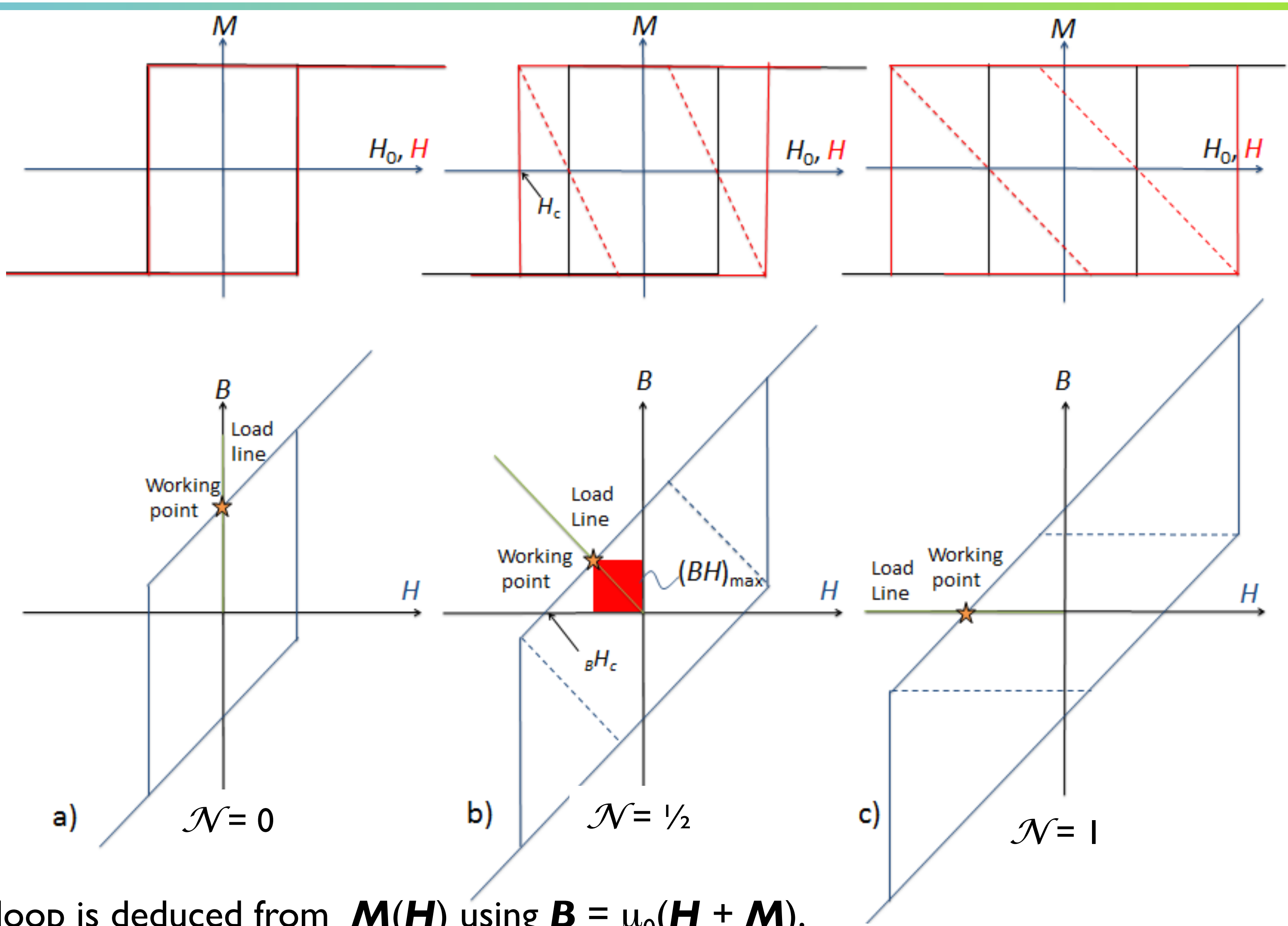
$\chi >> \chi'$ χ diverges as $T \rightarrow T_C$ but χ' never exceeds $1/\mathcal{N}$.







Demagnetizing correction to $M(H)$ and $B(H)$ loops.



The $\boldsymbol{B}(\boldsymbol{H})$ loop is deduced from $\boldsymbol{M}(\boldsymbol{H})$ using $\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$.

Permeability

In LIH media $\boxed{\boldsymbol{B} = \mu \boldsymbol{H}}$ defines the permeability μ Units: $TA^{-1}m$

Relative permeability is defined as $\mu_r = \mu / \mu_0$

$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$ gives $\boxed{\mu_r = 1 + \chi}$

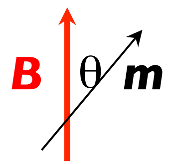
4. Energy and forces

Energy of ferromagnetic bodies

Torque and potential energy of a dipole in a field, assumed to be constant.

$$\boxed{\Gamma = \mathbf{m} \times \mathbf{B}} \qquad \boxed{\varepsilon = -\mathbf{m}.\mathbf{B}}$$

$$\Gamma = mB \sin\theta. \qquad \varepsilon = -mB \cos\theta.$$



Force

In a non-uniform field, $\boldsymbol{f} = -\nabla\varepsilon_m$ $\boxed{\boldsymbol{f} = \mathbf{m}.\nabla\mathbf{B}}$

- Magnetostatic (dipole-dipole) forces are long-ranged, but weak. They determine the magnetic microstructure (domains).

- $\frac{1}{2}\mu_0 H^2$ is the energy density associated with a magnetic field $\boldsymbol{H}$

M ≈ 1 MA m^{-1}, $\mu_0 H_d$ ≈ 1 T, hence $\mu_0 H_d M \approx 10^6$ J m^{-3}

- Products $\boldsymbol{B}.\boldsymbol{H}$, $\boldsymbol{B}.\boldsymbol{M}$, $\mu_0\boldsymbol{H}^2$, $\mu_0\boldsymbol{M}^2$ are all energies per unit volume.

- Magnetic forces *do no work* on moving charges $\boldsymbol{f} = q(\boldsymbol{v} \times \boldsymbol{B})$ [Lorentz force]

- No potential energy associated with the magnetic force.

Reciprocity theorem

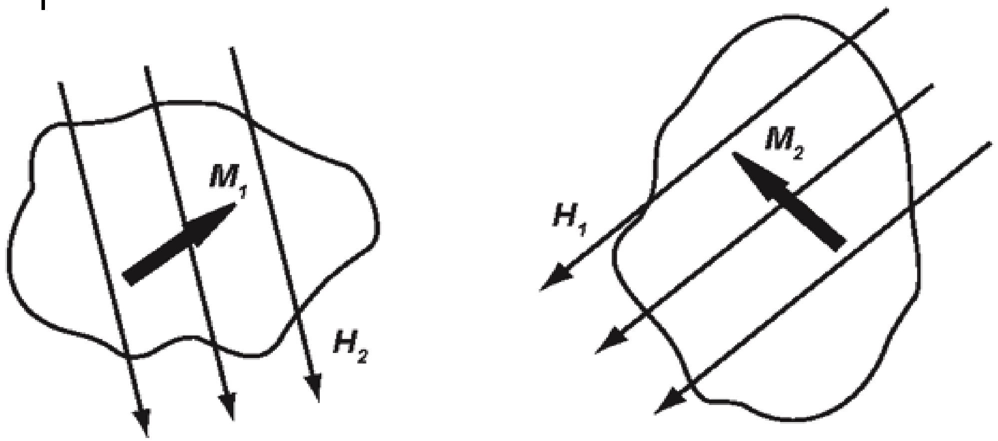
The interaction of a pair of dipoles, ε_p, can be considered as the energy of $\boldsymbol{m}_1$ in the field $\boldsymbol{B}_{21}$ created by $\boldsymbol{m}_2$ at $\boldsymbol{r}_1$ *or vice versa.*

$$\varepsilon_p = -\boldsymbol{m}_1 . \boldsymbol{B}_{21} = -\boldsymbol{m}_2 . \boldsymbol{B}_{12}$$

So $$\boxed{\varepsilon_p = -(1/2)(\boldsymbol{m}_1 . \boldsymbol{B}_{21} + \boldsymbol{m}_2 . \boldsymbol{B}_{12})}$$

Extending to magnetization distributions:

$$\varepsilon = -\mu_0 \int \boldsymbol{M}_1 . \boldsymbol{H}_2 \, d^3 r = -\mu_0 \int \boldsymbol{M}_2 . \boldsymbol{H}_1 \, d^3 r$$



Self Energy

The interaction of the body with the field it creates. $\boldsymbol{H}_d$.

Consider the energy to bring a small moment $\delta\boldsymbol{m}$ into position *within* the magnetized body

$$\delta\varepsilon = -\mu_0\, \delta\boldsymbol{m}.\boldsymbol{H}_{loc}$$

$$\boldsymbol{H}_{loc} = \boldsymbol{H}_d + (1/3)\boldsymbol{M}$$

Integration over the whole sample gives

$$\varepsilon = -\frac{1}{2}\int_v \mu_0 \boldsymbol{H}_d.\boldsymbol{M}\, d^3r - \frac{1}{6}\int_v \mu_0 M^2 d^3r.$$

The magnetostatic self energy is defined as

$$\boxed{\varepsilon_m = \varepsilon + (1/6)\int_v \mu_0 M^2 d^3r}$$

Or equivalently, using $\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$ and $\int \boldsymbol{B}.\boldsymbol{H}_d d^3r = 0$

$$\boxed{\varepsilon_m = \frac{1}{2}\int \mu_0 H_d^2 d^3r,}$$

For a uniformly magnetized ellipsoid

$$\varepsilon_m = \frac{1}{2}\mu_0 V \mathcal{N} M^2,$$

Energy associated with a field

General expression for the energy associated with a magnetic field distribution

$$\boxed{\varepsilon = \frac{1}{2}\int \mu_0 H^2 d^3r.}$$

Aim to maximize energy associated with the field created around the magnet, from previous slide:

$$\frac{1}{2}\int \mu_0 H_d^2 d^3r = -\frac{1}{2}\int_V \mu_0 \mathbf{H}_d.\mathbf{M}\, d^3r.$$

Can rewrite as:

$$\frac{1}{2}\int_o \mu_0 H_d^2 d^3r = -\frac{1}{2}\int_i \mu_0 H_d^2 d^3r - \frac{1}{2}\int_i \mu_0 \mathbf{M}.\mathbf{H}_d d^3r.$$

where we want to maximize the integral on the left.

Energy product: twice the energy stored in the stray field of the magnet

$$\boxed{-\mu_0 \int_i \boldsymbol{B}.\boldsymbol{H}_d \, d^3r}$$

Work done by an external field

Elemental work δw to produce a flux change $\delta\Phi$ is $I\,\delta\Phi$

Ampere: $\int \boldsymbol{H}.\mathrm{d}\boldsymbol{l} = I$ So $\delta w = \int \delta\Phi\, \boldsymbol{H}.\mathrm{d}\boldsymbol{l}$

So in general: $\delta w = \int \delta \boldsymbol{B}.\boldsymbol{H}.\mathrm{d}^3 r$

$\boldsymbol{H} = \boldsymbol{H}' + \boldsymbol{H}_d$ $\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$

Subtract the term associated with the H-field in empty space, to give the work done on the body by the external field;

$$\delta w' = \int (\boldsymbol{H}.\delta\boldsymbol{B} - \mu_0 \boldsymbol{H}'.\delta\boldsymbol{H}')d^3r.$$

$$\delta w' = \mu_0 \left[\int \delta(\boldsymbol{H}'.\boldsymbol{H}_d)d^3r + \int \boldsymbol{H}_d \delta \boldsymbol{H}_d d^3r + \int \boldsymbol{H}.\delta\boldsymbol{M}\, d^3r \right]$$

$$\delta w' = \delta\varepsilon_m + \mu_0 \int \boldsymbol{H}.\delta\boldsymbol{M}\ d^3r$$

$$\delta\varepsilon_m = -\frac{1}{2}\mu_0 \int (\boldsymbol{H}_d.\delta\boldsymbol{M} + \boldsymbol{M}.\delta\boldsymbol{H}_d)d^3r,$$

$$\delta\varepsilon_m = -\mu_0 \int \boldsymbol{H}_d.\delta\boldsymbol{M}\ d^3r,$$

gives

$$\boxed{\delta w' = \mu_0 \int \boldsymbol{H}'.\delta\boldsymbol{M}\ d^3r.}$$

Thermodynamics

First law: $dU = H_x dX + dQ$ $\qquad$ (U, Q, F, G are in units of Jm^{-3})

$dQ = TdS$

Four thermodynamic potentials;

$U(X,S)$

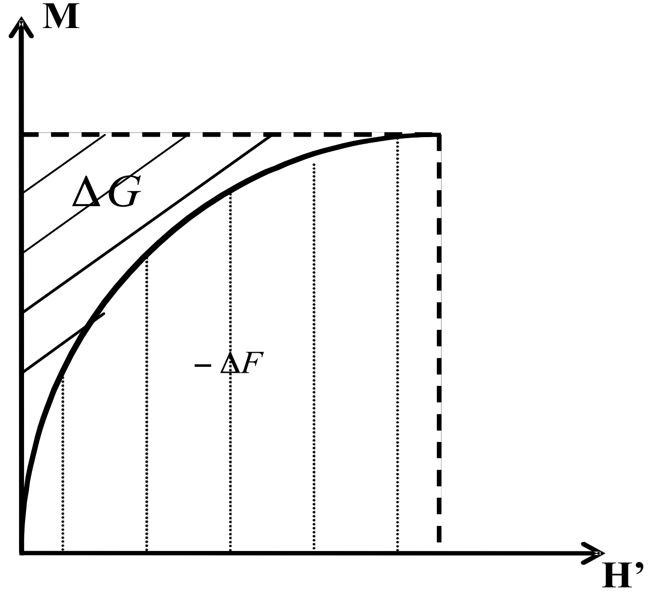
$E(H_X,S)$

$F(X,T) = U - TS \qquad dF = HdX - SdT$

$G(H_X,T) = F - H_X X \quad dG = -XdH - SdT$

Magnetic work is $H\delta B$ or $\mu_0 H'\delta M$

$$dF = \mu_0 H' dM - SdT$$
$$dG = -\mu_0 M dH' - SdT$$



$S = -(\partial G/\partial T)_{H'} \qquad \mu_0 M = -(\partial G/\partial H')_{T'}$

Maxwell relations

$(\partial S/\partial H')_{T'} = -\mu_0(\partial M/\partial T)_{H'}$ etc.

Magnetostatic forces

Force density on a magnetized body at constant temperature

$$\boxed{\boldsymbol{F}_m = -\nabla G}$$

$$\boldsymbol{F}_m = \nabla(\mu_0 \boldsymbol{H}'.\boldsymbol{M}) \qquad \nabla(\boldsymbol{H}'.\boldsymbol{M}) = (\boldsymbol{H}'.\nabla)\boldsymbol{M} + (\boldsymbol{M}.\nabla)\boldsymbol{H}'$$

Kelvin force

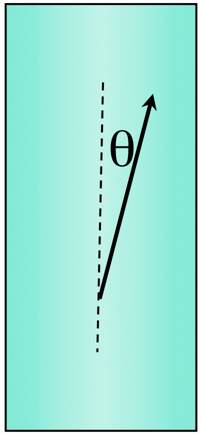
$$\boxed{\boldsymbol{F}_m = \mu_0(\boldsymbol{M}.\nabla)\boldsymbol{H}'}$$

General expression, for when ***M*** is dependent on ***H*** is

$$\boxed{\boldsymbol{F}_m = -\mu_0\nabla\left[\int_0^H \left(\frac{\partial M v}{\partial v}\right)_{H,T} dH\right] + \mu_0(\boldsymbol{M}.\nabla)\boldsymbol{H}.}$$

$v = 1/d$ d is the density

Anisotropy - shape



The energy of the system increases when the magnetization deviates by an angle θ from its easy axis.

$$E = K \sin^2 \theta$$

The anisotropy may be due to shape. The energy of the magnet in the demagnetizing field is $-(1/2)\mu_0 m H_d$

For shape anisotropy $K_{sh} = (1/4)\mu_0 M^2(1 - 3\mathcal{N})$

For example, if M = 1 MA m^{-1}, $\mathcal{N} = 1$, K_{sh} = 630 kJ m^{-3}

e.g. if a thin film is to have its magnetization perpendicular to the plane, there must be a stronger intrinsic magnetocrystalline anisotropy K_1.

Polarization and anisotropy field

Engineers often quote the *polarization* of a magnetic material, rather than its magnetization. The relation is simple; $J = \mu_0 M$.

For example, the polarization of iron is 2.15 T; Its magnetization is 1.71 MA m^{-1}

The defining relation is then: $\boxed{B = \mu_0 H + J}$

The anisotropy of whatever origin may be represented by an effective magnetic field, the *anisotropy field* H_a

$$H_a = 2K/\mu_0 M_s$$

The coercivity can never exceed the anisotropy field. In practice it is rarely possible to obtain coercivity greater than about 25 % of H_a. Coercivity depends on the microsctructure, and the ability to impede fprmation of reversed domains

Sometimes anisotropy field is quoted in Tesla. $B_a = \mu_0 H_a$.

Some expressions involving *B*

$\boldsymbol{F} = q(\boldsymbol{E} + \mathbf{v} \times \boldsymbol{B})$ Force on a charged particle *q*

$\boldsymbol{F} = \boldsymbol{B}\, I\ell$ Force on current-carrying wire

$\mathcal{E} = -\, d\Phi/dt$ Faraday's law of electromagnetic induction

$E = -\boldsymbol{m}.\boldsymbol{B}$ Energy of a magnetic moment

$\boldsymbol{F} = \nabla\, \boldsymbol{m}.\boldsymbol{B}$ Force on a magnetic moment

$\boldsymbol{\Gamma} = \boldsymbol{m} \times \boldsymbol{B}$ Torque on a magnetic moment

5. Units

A note on units:

Magnetism is an experimental science, and it is important to be able to calculate numerical values of the physical quantities involved. There is a strong case to use SI consistently

- SI units relate to the practical units of electricity measured on the multimeter and the oscilloscope
- It is possible to check the dimensions of any expression by inspection.
- They are almost universally used in teaching
- Units of ***B***, ***H***, Φ or $d\Phi/dt$ have been introduced.

BUT

Most literature still uses cgs units, You need to understand them too.

SI / cgs conversions:

	SI units		cgs units	
	$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$		$\boldsymbol{B} = \boldsymbol{H} + 4\pi\boldsymbol{M}$	
m	A m^2		emu	
$\boldsymbol{M}$	A m^{-1}	(10^{-3} emu cc^{-1})	emu cc^{-1}	(1 k A m^{-1})
σ	A m^2 kg^{-1}	(1 emu g^{-1})	emu g^{-1}	(1 A m^2 kg^{-1})
$\boldsymbol{H}$	A m^{-1}	(4π/1000 ≈ 0.0125 Oe)	Oersted	(1000/4π ≈ 80 A m^{-1})
$\boldsymbol{B}$	Tesla	(10000 G)	Gauss	(10^{-4} T)
Φ	Weber (Tm^2)	(10^8 Mw)	Maxwell (G cm^2)	(10^{-8} Wb)
$d\Phi/dt$	V	(10^8 Mw s^{-1})	Mw s^{-1}	(10 nV)
χ	-	(4π cgs)	-	(1/4π SI)

Mechanical

Quantity	Symbol	Unit	m	l	t	i	θ
Area	$\mathcal{A}$	m^2	0	2	0	0	0
Volume	V	m^3	0	3	0	0	0
Velocity	v	$m\ s^{-1}$	0	1	−1	0	0
Acceleration	a	$m\ s^{-2}$	0	1	−2	0	0
Density	d	$kg\ m^{-3}$	1	−3	0	0	0
Energy	ε	J	1	2	−2	0	0
Momentum	p	$kg\ m\ s^{-1}$	1	1	−1	0	0
Angular momentum	L	$kg\ m^2\ s^{-1}$	1	2	−1	0	0
Moment of inertia	I	$kg\ m^2$	1	2	0	0	0
Force	f	N	1	1	−2	0	0
Force density	F	$N\ m^{-3}$	1	−2	−2	0	0
Power	P	W	1	2	−3	0	0
Pressure	P	Pa	1	−1	−2	0	0
Stress	σ	$N\ m^{-2}$	1	−1	−2	0	0
Elastic modulus	K	$N\ m^{-2}$	1	−1	−2	0	0
Frequency	f	s^{-1}	0	0	−1	0	0
Diffusion coefficient	D	$m^2\ s^{-1}$	0	2	−1	0	0
Viscosity (dynamic)	η	$N\ s\ m^{-2}$	1	−1	−1	0	0
Viscosity	ν	$m^2\ s^{-1}$	0	2	−1	0	0
Planck's constant	$\hbar$	J s	1	2	−1	0	0

Electrical

Quantity	Symbol	Unit	m	l	t	i	θ
Current	I	A	0	0	0	1	0
Current density	j	$\mathrm{A\,m^{-2}}$	0	−2	0	1	0
Charge	q	C	0	0	1	1	0
Potential	V	V	1	2	−3	−1	0
Electromotive force	$\mathcal{E}$	V	1	2	−3	−1	0
Capacitance	C	F	−1	−2	4	2	0
Resistance	R	Ω	1	2	−3	−2	0
Resistivity	ϱ	$\Omega\,\mathrm{m}$	1	3	−3	−2	0
Conductivity	σ	$\mathrm{S\,m^{-1}}$	−1	−3	3	2	0
Dipole moment	p	C m	0	1	1	1	0
Electric polarization	P	$\mathrm{C\,m^{-2}}$	0	−2	1	1	0
Electric field	E	$\mathrm{V\,m^{-1}}$	1	1	−3	−1	0
Electric displacement	D	$\mathrm{C\,m^{-2}}$	0	−2	1	1	0
Electric flux	Ψ	C	0	0	1	1	0
Permittivity	ε	$\mathrm{F\,m^{-1}}$	−1	−3	4	2	0
Thermopower	S	$\mathrm{V\,K^{-1}}$	1	2	−3	−1	−1
Mobility	μ	$\mathrm{m^2\,V^{-1}\,s^{-1}}$	−1	0	2	1	0

Magnetic

Quantity	Symbol	Unit	m	l	t	i	θ
Magnetic moment	$\mathfrak{m}$	A m^2	0	2	0	1	0
Magnetization	M	A m^{-1}	0	−1	0	1	0
Specific moment	σ	A m^2 kg^{-1}	−1	2	0	1	0
Magnetic field strength	H	A m^{-1}	0	−1	0	1	0
Magnetic flux	Φ	Wb	1	2	−2	−1	0
Magnetic flux density	B	T	1	0	−2	−1	0
Inductance	L	H	1	2	−2	−2	0
Susceptibility (M/H)	χ		0	0	0	0	0
Permeability (B/H)	μ	H m^{-1}	1	1	−2	−2	0
Magnetic polarization	J	T	1	0	−2	−1	0
Magnetomotive force	$\mathcal{F}$	A	0	0	0	1	0
Magnetic ‘charge’	q_m	A m	0	1	0	1	0
Energy product	(BH)	J m^{-3}	1	−1	−2	0	0
Anisotropy energy	K	J m^{-3}	1	−1	−2	0	0
Exchange stiffness	A	J m^{-1}	1	1	−2	0	0
Hall coefficient	R_H	m^3 C^{-1}	0	3	−1	−1	0
Scalar potential	φ	A	0	0	0	1	0
Vector potential	A	T m	1	1	−2	−1	0
Permeance	P_m	T m^2 A^{-1}	1	2	−2	−2	0
Reluctance	R_m	A T^{-1} m^{-2}	−1	−2	2	2	0

Thermal

Quantity	Symbol	Unit	m	l	t	i	θ
Enthalpy	H	J	1	2	−2	0	0
Entropy	S	J K^{-1}	1	2	−2	0	−1
Specific heat	C	J K^{-1} kg^{-1}	0	2	−2	0	−1
Heat capacity	c	J K^{-1}	1	2	−2	0	−1
Thermal conductivity	κ	W m^{-1} K^{-1}	1	1	−3	0	−1
Sommerfeld coefficient	γ	J mol^{-1} K^{-1}	1	2	−2	0	−1
Boltzmann's constant	k_B	J K^{-1}	1	2	−2	0	−1

(1) Kinetic energy of a body: $\varepsilon = \frac{1}{2}mv^2$

$[\varepsilon] = [1, 2, -2, 0, 0]$

$[m] = [1, 0, 0, 0, 0]$

$[v^2] = \dfrac{2[0, -1, -1, 0, 0]}{[1, -2, -2, 0, 0]}$

(2) Lorentz force on a moving charge; $\boldsymbol{f} = q\boldsymbol{v} \times \boldsymbol{B}$

$[f] = [1, 1, -2, 0, 0]$

$[q] = [0, 0, 1, 1, 0]$

$[v] = [0, 1, -1, 0, 0]$

$[B] = \dfrac{[1, 0, -2, -1, 0]}{[1, 1, -2, 0, 0]}$

(3) Domain wall energy $\gamma_w = \sqrt{AK}$ (γ_w is an energy per unit area)

$[\gamma_w] = [\varepsilon A^{-1}]$

$= [1, 2, -2, 0, 0]$

$-\ [0, 2, 0, 0, 0]$

$= [1, 0, -2, 0, 0]$

$[\sqrt{AK}] = 1/2[AK]$

$[\sqrt{A}] = \frac{1}{2}[1, 1, -2, 0, 0]$

$[\sqrt{K}] = \frac{1}{2}\dfrac{[1, -1, -2, 0, 0]}{[1, 0, -2, 0, 0]}$

(4) Magnetohydrodynamic force on a moving conductor $\boldsymbol{F} = \sigma \boldsymbol{v} \times \boldsymbol{B} \times \boldsymbol{B}$ ($\boldsymbol{F}$ is a force per unit volume)

$$[F] = [FV^{-1}] = [1, 1, -2, 0, 0] - \frac{[0, 3, 0, 0, 0]}{[1, -2, -2, 0, 0]}$$

$$[\sigma] = [-1, -3, 3, 2, 0]$$
$$[v] = [0, 1, -1, 0, 0]$$
$$[B^2] = \frac{2[1, 0, -2, -1, 0]}{[1, -2, -2, 0, 0]}$$

(5) Flux density in a solid $\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$ (note that quantities added or subtracted in a bracket must have the same dimensions)

$$[B] = [1, 0, -2, -1, 0]$$

$$[\mu_0] = [1, 1, -2, -2, 0]$$
$$[M], [H] = \frac{[0, -1, 0, 1, 0]}{[1, 0, -2, -1, 0]}$$

(6) Maxwell's equation $\nabla \times \boldsymbol{H} = \boldsymbol{j} + \mathrm{d}\boldsymbol{D}/\mathrm{d}t$.

$$[\nabla \times \boldsymbol{H}] = [Hr^{-1}] = [0, -1, 0, 1, 0] - [0, 1, 0, 0, 0] = [0, -2, 0, 1, 0]$$

$$[j] = [0, -2, 0, 1, 0]$$

$$[\mathrm{d}\boldsymbol{D}/\mathrm{d}t] = [Dt^{-1}] = [0, -2, 1, 1, 0] - [0, 0, 1, 0, 0] = [0, -2, 0, 1, 0]$$

(7) Ohm's Law $V = IR$

$$= [1, 2, -3, -1, 0]$$

$$[0, 0, 0, 1, 0] + [1, 2, -3, -2, 0] = [1, 2, -3, -1, 0]$$

(8) Faraday's Law $\mathcal{E} = -\partial\Phi/\partial t$

$$= [1, 2, -3, -1, 0]$$

$$[1, 2, -2, -1, 0] - [0, 0, 1, 0, 0] = [1, 2, -3, -1, 0]$$