A few analytical micromagnetic problems

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(Dated: 28 août 2011)

I. FRAMEWORK OF MICROMAGNETISM

Most of micromagnetic modeling relies on two hypotheses:

- The spatial variation of any physical quantity (magnetic moments etc.) is slow at the length scale of inter-atomic distances. This allows one to describe physical systems in a continuous medium approach and make use of the power of integral theory and differential equations.
- The resulting magnetization vector field (i.e. the density of magnetic moments per unit volume) has a uniform and constant magnitude : $|\mathbf{M}(\mathbf{r})| \equiv M_s$, the spontaneous magnetization.

A major purpose of micromagnetism is to exhibit stable (or metastable) magnetization arrangements under static conditions. This minimizes globally (resp. locally) the total energy of the system.

In most situations the density of energy comprises at most four terms : magnetic anisotropy $E_a = K f_a(\theta,\varphi)$, Zeeman energy $E_Z = -\mu_0 \mathbf{M}.\mathbf{H}$, self-dipolar energy $E_d = -\frac{1}{2}\mu_0 \mathbf{M}.\mathbf{H_d}$ and exchange energy, which continuous form we propose to link with microscopic quantities in this paragraph.

Let us consider exchange energy in a Heisenberg model : $\mathcal{E} = -\frac{1}{2}\sum_{<i,j>} J\mathbf{S}_i.\mathbf{S}_j$, where the summation concerns near(est) neighbors pairs $< i,j >$. $J > 0$ for ferromagnets.

In the simple framework of a one-dimensional crystal with atomic spacing a, the energy reads

$$\begin{aligned}
\mathcal{E} &= -\sum_i J\mathbf{S}_i.\mathbf{S}_{i+1} \qquad (1)\\
&= -\sum_i J\mathbf{S}(ia).\mathbf{S}((i+1)a)
\end{aligned}$$

If $\mathbf{S}(x)$ varies slowly on the scale of a, $\mathbf{S}((i+1)a) \approx \mathbf{S}(ia) + a\partial_x\mathbf{S}(ia) + \frac{1}{2}a^2\partial_x^2\mathbf{S}(ia)$. This expression can also be expressed in term of $\theta(x)$ the angle between $\mathbf{S}(x)$ and the x axis:

$$\begin{aligned}
\mathbf{S}(x) &= [\cos\theta(x), \sin\theta(x)]\\
\partial_x\mathbf{S}(x) &= [-\partial_x\theta(x)\sin\theta(x), \partial_x\theta(x)\cos\theta(x)]\\
\partial_x^2\mathbf{S}(x) &= [-\partial_x^2\theta(x)\sin\theta(x) - (\partial_x\theta(x))^2\cos\theta(x), \partial_x^2\theta\cos\theta(x) - (\partial_x\theta)^2\sin\theta(x)]
\end{aligned}$$

Then,

$$\begin{aligned}
\mathcal{E} &= \sum_i \frac{Ja^2}{2}[\partial_x\theta(ia)]^2\\
&\approx \int \frac{Ja}{2}[\partial_x\theta(x)]^2 dx
\end{aligned}$$

The exchange energy density is then

$$E_{ex} = A(\partial_x\theta)^2 \qquad (2)$$

with $A = Ja/2$ the exchange constant.

In a three-dimensional body this energy is generalized to the expression :

$$E_{ex} = A(\nabla\mathbf{m})^2 \qquad (3)$$

where $(\nabla\mathbf{m})^2$ is a shortcut for $\sum_i\sum_j(\partial_{x_j}m_i)^2$. The exchange constant is $A \propto J$ in 2d and $A \propto J/a$ in 3d, the proportionality constant depending on the type of atomic lattice.

II. EULER-LAGRANGE EQUATION

We will seek to exhibit a magnetization configuration that minimizes the energy density integrated over the entire system : $\mathcal{E} = \int E(\mathbf{r})d\mathbf{r}$. The problem of finding the minimum of a continuous quantity integrated over space is a common problem solved through Euler-Lagrange equation, which we will deal with in a textbook one-dimensional framework here.

Let us consider a microscopic quantity defined as $F(\theta; d_x\theta)$, where x is the spatial coordinate and θ a quantity defined at each point. In the case of micromagnetism F is in general the energy:

$$F(\theta; d_x\theta) = A(\partial_x\theta)^2 + f(\theta)$$

We define the integrated quantity

$$\mathcal{F} = \int_A^B F(\theta; d_x\theta)dx + E_A(\theta) + E_B(\theta)$$

where A and B are the boundaries and $E_A(\theta)$ and $E_B(\theta)$ are surface energy terms.

Let us consider a solution θ which minimizes $\mathcal{F}$ and an infinitesimal function variation $\delta\theta(x)$ of θ.

$$\begin{aligned}
\mathcal{F}(\theta + \delta\theta) &= \int_A^B F(\theta + \delta\theta; d_x(\theta + \delta\theta))dx + E_A(\theta + \delta\theta) + E_B(\theta + \delta\theta) \\
&\approx \int_A^B dx \left\{ F(\theta; d_x\theta) + \delta\theta \frac{\partial F}{\partial\theta}(\theta; d_x\theta) + d_x\delta\theta \frac{\partial F}{\partial d_x\theta}(\theta; d_x\theta) \right\} + \\
&\qquad E_A(\theta) + E_B(\theta) + \delta\theta \left\{ \frac{\partial E_A}{\partial\theta}(\delta\theta) + \frac{\partial E_B}{\partial\theta}(\delta\theta) \right\}
\end{aligned}$$

If θ is a solution then $\Delta\mathcal{F} = \mathcal{F}(\theta + \delta\theta) - \mathcal{F}(\theta) = 0$:

$$\Delta\mathcal{F} = \int_A^B dx \left\{ \delta\theta \frac{\partial F}{\partial\theta}(\theta; d_x\theta) + d_x\delta\theta \frac{\partial F}{\partial d_x\theta}(\theta; d_x\theta) \right\} + \delta\theta \left\{ \frac{\partial E_A}{\partial\theta}(\delta\theta) + \frac{\partial E_B}{\partial\theta}(\delta\theta) \right\}$$

The second term in the integral can be simplified using a partial integration:

$$\int_A^B dx \;\; d_x\delta\theta \frac{\partial F}{\partial d_x\theta}(\theta; d_x\theta) = \left| \delta\theta \frac{\partial F}{\partial d_x\theta}(\theta; d_x\theta) \right|_B^A - \int_A^B dx \;\; \delta\theta \frac{\partial^2 F}{\partial x \partial d_x\theta}(\theta; d_x\theta)$$

We the obtain the Euler-Lagrange equation :

$$\partial_\theta F - \partial_x(\partial_{\partial_x\theta} F) = 0 \tag{4}$$
$$\partial_\theta E_A - \partial_{\partial_x\theta} F|_A = 0 \tag{5}$$
$$\partial_\theta E_B + \partial_{\partial_x\theta} F|_B = 0 \tag{6}$$

Notice that equations Eq. 5 and Eq. 6 differ in sign because a surface quantity should be defined with respect to the unit vector normal to the surface, with a unique convention for the sense, such as the outwards normal. Here the abscissa x is outwards at point B however inwards at A. An alternative microscopic explanation would be that for a given sign of $d_x\theta$ the exchange torque exerted on a moment to the right (at point B) is opposite to that exerted to the left (at point A), whereas the torque exerted by a surface anisotropy energy solely depends on θ.

III. BLOCH DOMAIN WALL

We consider a infinite 1d system, with exchange A and a magnetic anisotropy of the simplest form (uniaxial and second order) : $E_a(\theta) = K\sin^2\theta$. Let us apply the exquation 4-6 to this system. We consider free boundary condition so that $E_A = E_B = 0$. The Euler-Lagrange equation becomes:

$$2A\partial_x^2\theta = 2K\sin\theta\cos\theta \tag{7}$$
$$2A\partial_x\theta|_{+\infty} = 0 \tag{8}$$
$$2A\partial_x\theta|_{-\infty} = 0 \tag{9}$$

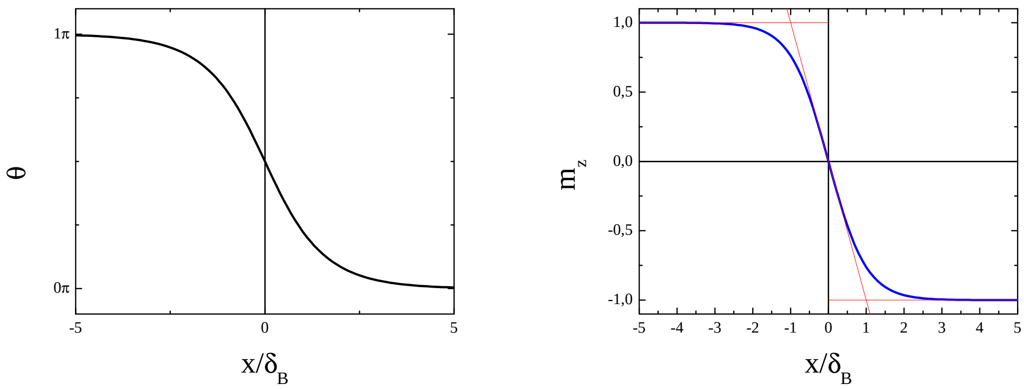

FIG. 1 – *Variation of the out of plane angle of the magnetization θ and the out of plane magnetization component m_z for a Bloch domain wall. The red thin lines represent the asymptotes approximation.*

The two last equations mean that the derivative at $+$ or $-\infty$ are zero, so that $\theta(x)$ cannot diverge.

There are 4 trivial solutions: $\theta(x) = 0$, $\pi/2$, π, and $3\pi/2$, which are uniform. However, they are not all stable, 0 and π being stable (the anisotropy energy density is zero), and $\pi/2$ and $3\pi/2$ being unstable (the anisotropy energy density is K). We now look for a non uniform solution, which forms a domain wall, i.e. where $\theta(-\infty) = 0$ and $\theta(+\infty) = \pi$.

Let us multiply equation 7 by $\partial_x\theta$ (different from zero for a non uniform solution) and integrate:

$$\begin{aligned} 2A\partial_x\theta\partial_x^2\theta &= 2K\partial_x\theta\sin\theta\cos\theta \\ A(\partial_x\theta)^2 &= K\sin^2\theta \end{aligned}$$

We obtain :

$$\frac{d\theta}{\sin\theta} = \pm\frac{dx}{\delta_B} \tag{10}$$

with $\delta_B = \sqrt{A/K}$.

We use the variable $t = \tan\theta/2$ and remark that $\sin\theta = 2t/(1+t^2)$ and $dt = \frac{1}{2}d\theta(1+\tan^2(\theta/2)) = \frac{1}{2}d\theta(1+t^2)$. Equation 10 becomes:

$$\frac{dt}{t} = \pm\frac{dx}{\delta_B}$$

which has the simple solution

$$\begin{aligned} \frac{x}{\delta_B} &= \pm\ln(t) + \frac{x_0}{\delta_B} \\ \text{or} \quad \theta &= 2\arctan\left[\exp\left(\pm\frac{x-x_0}{\delta_B}\right)\right] \end{aligned} \tag{11}$$

or equivalently, for m_z[1]:

$$m_z(x) = \pm\tanh(x/\delta_B) \tag{12}$$

This solution corresponds to a domain wall at x_0. Given the signe $+$ or $-$ the magnetization turns from $\theta = 0$ to π ($+$ sign) or from π to 0. All these solutions are equivalent. The Bloch wall parameter δ_B is proportionnal to the domain wall width. It corresponds to one of the characteristic length in micromagnetism, which compares the magnetic anisotropy strength with the exchange energy. It is easily seen that higher anisotropy energy favors narrower domain walls.

Let us now calculate the energy of the domain wall. It corresponds to the difference between the energy in the presence and in the abscence of a domain wall. This last energy being 0 we have :

$$\sigma = \int dx \left[A(\partial_x \theta)^2 + K \sin^2 \theta \right]$$

with θ being given by eq. 11

$$\begin{aligned} \sin[\theta(x)] &= \frac{1}{\cosh(x/\delta_B)} \\ \partial_x \theta &= \frac{1}{\delta_B \cosh(x/\delta_B)} \end{aligned}$$

$$\begin{aligned} \sigma &= \int \frac{dx}{\delta_B^2} \left[\frac{A}{\cosh^2(x/\delta_B)} + \frac{K\delta_B^2}{\cosh^2(x/\delta_B)} \right] \\ &= 2K\delta_B \int du \left[\frac{1}{\cosh^2(u)} \right] \text{ with } u = x/\delta_B \end{aligned}$$

$$\sigma = 4\sqrt{AK} \tag{13}$$

IV. AN EXAMPLE OF PINNING

Starting from a homogeneous material let us model a local defect in the form of a magnetically-soft (i.e. zero anisotropy) insertion of width δl, located at position x. In the case where $\delta\ell \ll \delta_B$ we consider that the domain wall profile is not deformed as compared to the defect free problem. The energy of the system is then

$$\begin{aligned} \mathcal{E} &= 2\sqrt{AK} \left[\int_{-\infty}^{\frac{x}{\delta_B} - \frac{\delta\ell}{2\delta_B}} \frac{1}{\cosh^2 u} + \int_{\frac{x}{\delta_B} + \frac{\delta\ell}{2\delta_B}}^{+\infty} \frac{1}{\cosh^2 u} \right] \\ &= 4\sqrt{AK} \left[1 + \frac{1}{2} \tanh\left(\frac{x}{\delta_B} - \frac{\delta\ell}{2\delta_B} \right) - \frac{1}{2} \tanh\left(\frac{x}{\delta_B} + \frac{\delta\ell}{2\delta_B} \right) \right] \\ &\approx 4\sqrt{AK} \left[1 - \frac{\delta\ell}{4\delta_B} \frac{1}{\cosh^2(x/\delta_B)} \right] \\ &\approx \sigma \left[1 - \frac{\delta\ell}{4\delta_B} \frac{1}{\cosh^2(x/\delta_B)} \right] \end{aligned}$$

When a magnetic field is applied, the Zeeman energy varies when the domain wall moves so that Zeeman energy is proportionnal to the domain wall position:

$$\mathcal{E}_Z = -2x\mu_0 M_S H \cos\alpha. \tag{14}$$

The minus sign is just conventionnal and tells that a positive field tends to push the domain wall toward positive x (the opposite convention can also be choosen). The 2 tells that when the domain wall moves, the local magnetization endergoes a π angle rotation. Note that, implicitly, we have neglected the magnetization rotation induced by the field angle outside of the domain wall, which is only valid for low field. When the magnetic field is applied, the potential is tilted and the depinning field is reached when the slope is always negative.

The slope is given by :

$$\frac{d\mathcal{E}}{dx} = \frac{\sigma\delta\ell}{2\delta_B^2} \frac{\sinh(x/\delta_B)}{\cosh^3(x/\delta_B)} - 2\mu_0 M_S H \cos\alpha \tag{15}$$

The maximum slope is obtained at the inflexion point, for which $\partial^2\mathcal{E} = 0$:

$$\frac{d^2\mathcal{E}}{dx^2} = \frac{\sigma\delta\ell}{2\delta_B^3} \frac{\cosh^2(x/\delta_B) - 3\sinh^2(x/\delta_B)}{\cosh^4(x/\delta_B)} \tag{16}$$

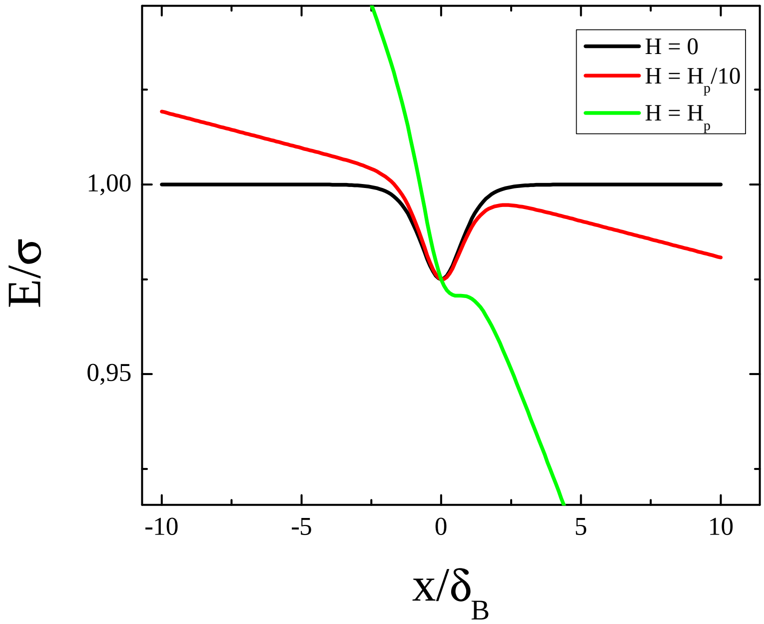

FIG. 2 – *Variation of the domain wall energy as a function of its position when the defect is in* $x = 0$ *and for different value of the magnetic field.*

We deduce that at the inflexion point,

$$\begin{aligned}\sinh(x/\delta_B) &= 1/\sqrt{2} \\ \cosh(x/\delta_B) &= \sqrt{3/2}\end{aligned} \tag{17}$$

At the depinning field, the slope of the potential should change sign at the inflexion point. H_p is thus determined as

$$\left.\frac{d\mathcal{E}}{dx}\right|_{\text{infl.point.}} = \frac{\sigma\delta\ell}{2\delta_B^2}\frac{\sinh(x/\delta_B)}{\cosh^3(x/\delta_B)} - 2\mu_0 M_S H_p \cos\alpha = 0$$

So that

$$\begin{aligned}H_p &= \frac{\sigma\delta\ell}{2\delta_B^2\mu_0 M_S\cos\alpha}\frac{1}{3\sqrt{3}} \\ &= \frac{H_K}{\cos\alpha}\frac{\delta\ell}{\delta_B}\frac{1}{3\sqrt{3}}\end{aligned} \tag{18}$$

with $H_K = 2K/\mu_0 M_S = \sigma/(2\delta_B\mu_0 M_S)$.

Notice :

- The model of the Bloch wall was named after D. Bloch who published this model in 1932[2].
- The $1/\cos\alpha$ dependence of coercivity comes from the weak field hypothesis ($H_p \ll H_K$), which occurs for low pinning. It is known as the Kondorski model[3].
- This model had been initially published in 1939 by Becker and Döring[4], and is summarized in the nice book of Skomsky Simple models of Magnetism[5].
- While coercivity requires a high anisotropy, the latter is not a suficient condition to have a high coercivity. To achieve this one must prevent magnetization reversal that can be initiated on defects (structural or geometric) and switch the entire magnetization by propagation of a domain wall. In a short-hand classification one distinguishes coercivity made possible by hindering nucleation, or hindering the propagation of domain walls. In reality both phenomena are often intermixed. Here we modeled an example of pinning.

– Simple micromagnetic models of nucleation on defects[6] were the first to be exhibited to tentatively explain the so-called Brown paradox, i.e. the fact that values of experimental values of coercivity in most samples are smaller or much smaller than the values predicted by the ideal model of coherent rotation[7].

V. MAGNETIC VORTEX IN A NANODOT

This last exemple intend to present an other emblematic problem of micromangetism, the magnetic vortex. It is particularly interesting to introduce the exchange length Λ, which is another characteristic length of micromagnetism.

We consider a magnetic nanodisk made of a soft magnetic material (ie. magnetocrystalline anisotropy is neglected). In order to minimize the dipolar energy, the magnetization tends to close the magnetic flux and form a vortex state: the magnetization then turns around the disk center. This effect will occur for nanodisk with diamter larger than several exchange length[8].

In this problem, we present a shematic calculation of the vortex configuration, in the case of ultrathin nanodisk (thiner than the exchange length), for which the magnetization is uniform accros the dot thickness.

A. 2D problem

We first consider that the magnetization lies in the disk plane. Indeed, this avoids the creation of magnetic charges on the top and bottom faces of the disk, which would result in a dipolar energy cost. In polar (r,φ) coordinates, the magnetization writes:

$$\mathbf{m} = [-\sin(\varphi); \cos(\varphi); 0] \,. \tag{19}$$

This solution ensures $div(\mathbf{m}) = 0$ and $\mathbf{m}.\mathbf{n} = 0$ (no surface charges). This means that the dipolar energy is perfectly satified and the energy cost is zero. Only the exchange energy remains:

$$\begin{aligned} E_{ex} &= A(\nabla \mathbf{m})^2 \\ &= (\partial_x m_x)^2 + (\partial_y mx)^2 + (\partial_x m_y)^2 + (\partial_y my)^2 \\ &= A/r^2. \end{aligned} \tag{20}$$

Obviously, the exchange energy diverges at the disk center, which is not possible in reality. The problem comes from the hypothesis that the magnetization is forced to lie in the disk plane.

B. 3D problem

To avoid the divergence of the exchange energy at the disk center, we let the possibility for the magnetization is explore the direction perpendicular to the disk plane. Using $\theta(r)$, the angle of the magnetization with the disk normal, and keeping the cirular geometry, the magnetization now writes

$$\mathbf{m}(r,\varphi) = [-\sin(\varphi)\sin(\theta); \cos(\varphi)\sin(\theta); \cos(\theta)] \,. \tag{21}$$

The exchange energy is then

$$\begin{aligned} E_{ex} &= A(\nabla \mathbf{m})^2 \\ &= A\left[\frac{\sin^2\theta}{r^2} + \left(\frac{d\theta}{dr}\right)^2\right] \end{aligned} \tag{22}$$

Obviously, in order to avoid the exchange energy divergence, $\theta(r = 0) = 0$, which means that the magnetization is perpendicular to the disk plane at the disk center. This corresponds to the vortex core. The purpose of the following is to estimate the width of this vortex core.

However, the condition **m.n** is no longer satisfied and thus, the m_z component results in magnetic charges on the surfaces of the disk. This results in a dipolar energy cost :

$$E_{dip} = \frac{1}{2}\mu_0 M_S^2 \cos(\theta) \tag{23}$$

The overall energy density reads:

$$\mathcal{E} = 2\pi t \int_0^R dr\ r \left[A\left(\frac{d\theta}{dr}\right)^2 + A\frac{\sin^2\theta}{r^2} + \frac{\mu_0 M_s^2}{2}\cos^2\theta \right] \tag{24}$$

This problem is then equivalent to a 1d problem with $r \in [0; R]$ as it has been treated before. We apply the Euler-Lagrange equations (beware not that the term inside the integral is not only the local energy as in the case of the Bloch wall, but that a r multiplier has to be taken into account due to the circular geometry):

$$\left(\frac{1}{r^2} - \frac{1}{\Lambda^2}\right)\sin 2\theta = \frac{1}{r}\frac{d(2\theta)}{dr} + \frac{\partial^2(2\theta)}{\partial r^2} \tag{25}$$

$$2Ar\frac{d\theta}{dr}\bigg|_{r=0} = 0 \tag{26}$$

$$2Ar\frac{d\theta}{dr}\bigg|_{r=R} = 0 \tag{27}$$

with $\Lambda = \sqrt{2A/\mu_0 M_s^2}$ the so-called exchange length. This length is the other characterisitc length in micromagnetism. It compares the dipolar energy strength to the exchange energy. Below this length, exchange energy dominates and dipolar effect can be neglected.

Equation 26 is always satisfied what ever $\theta(r)$. Equation 27 means that $\partial_r\theta|_R = 0$.

Unfortunately, no analytical solution can be found to equation 25, which has to be solved numerically. For this purpose, we use the "shooting method": starting with $r = 0$ and $\theta(r = 0) = 0$, we need to know $\partial_r\theta(r = 0)$ ("shooting angle"). We integrate equation 25, with various values for $\partial_r\theta(r = 0)$ (using for example Runge Kutta method) and only conserve the solution if $\partial_r\theta(r = R) = 0$. The solution is not unique and a large range of solution can satisfy equation 27. To discriminate between them, the energy (equation 24) is calculated. The solution is shown in fig. 3.

Finaly, the vortex core width is found to be of the order of Λ (taking the width at half amplitude of the peak for m_z, we obtain 2.2Λ).

This model has first been presented by Feldkeller and Thomas[9]. Although it is not anlytical up to the end, its importance relies in the introduction of the exchange length.

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[1] using the half angle formula:
$t = \tan(A/2)$
$\sin A = 2t/(1 + t^2)$, $\cos A = (1 - t^2)/(1 + t^2)$

[2] F. Bloch, Z. Phys. 74, 295 (1932).
[3] E. Kondorski, On the nature of coercive force and irreversible changes in magnetisation, Phys. Z. Sowjetunion 11, 597 (1937).
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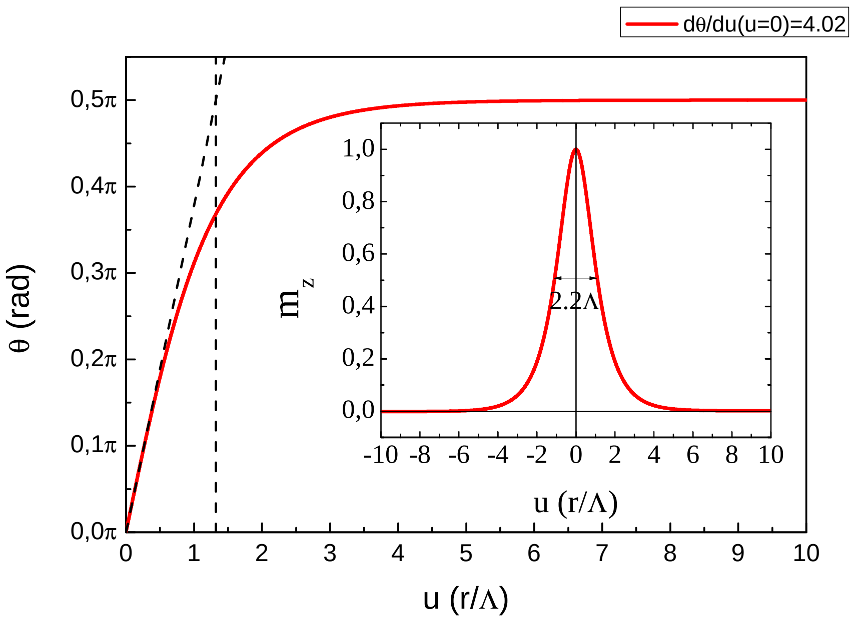

FIG. 3 – *Variation of the out of plane angle of the magnetization θ along the radius of the nanodot. Inset: variation of the $m_z = \cos\theta$. The result has been obtained by integrating eq. 25 with $R = 10\Lambda$ and* $d\theta/du(u = 0) = 4.02$.