Nanomagnetism – Practical on units – Answers

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2 Expressing dimensions

The principle for all derivations is the same: we must exhibit any formula linking the variable for which we wish to determine the dimension, with variables already processed, starting with the four elementary dimensions.

2.1 Mechanics

- Force may be related to inertial mass and acceleration: $\mathbf{F} = m\mathbf{a}$. Thus, we have $[F] = [m] + [a] = [0\ 1\ 0\ 0] + [1\ 0\ -2\ 0]$, so in the end: $[\mathbf{F}] = [1\ 1\ -2\ 0]$

- Energy $\mathcal{E}$ may result from the work of a force: $\mathcal{E} = \mathbf{F}.\boldsymbol{\ell}$, where $\boldsymbol{\ell}$ is a distance vector. Thus, $[\mathcal{E}] = [\mathbf{F}] + [\mathrm{L}] = [1\ 1\ -2\ 0] + [1\ 0\ 0\ 0]$. In the end: $[\mathcal{E}] = [2\ 1\ -2\ 0]$.

- Power is an energy per unit time: $\mathcal{P} = \mathcal{E}/t$, thus: $[\mathcal{P}] = [2\ 1\ -3\ 0]$.

- The volume density of energy is $E = \mathcal{E}/V$ where V is a volume, so: $[E] = [-1\ 1\ -2\ 0]$.

- The volume density of power is $P = \mathcal{P}/V$, so: $[P] = [-1\ 1\ -3\ 0]$.

2.2 Electricity

- The electric field $\mathbf{E}$ may be related to mechanics through a force $\mathbf{F}$ acting on a charge q: $\mathbf{F} = q\mathbf{E}$. Thus $[\mathbf{E}] = [\mathbf{F}] - [q] = [\mathbf{F}] - ([\mathrm{A}] + [\mathrm{T}])$, as the charge is current times the time. In the end: $[\mathbf{E}] = [1\ 1\ -3\ -1]$. Other formula may be used, such as $P = UI$ for an electric circuit, with U a voltage, linked to an electric field with $U = E\ell$.

- Voltage U is related to the electric field with: $U = E\ell$. Thus: $[U] = [2\ 1\ -3\ -1]$.

- Resistance may be found with $U = RI$, yielding: $[R] = [2\ 1\ -3\ -2]$.

- Resistivity may be related to the resistance of a wire through: $R = \rho\ell/S$ with S its cross-sectional area. Thus: $[\rho] = [3\ 1\ -3\ -2]$.

- The permittivity relates electric charges, to their consequence on fields, which are a representation of quantities affecting mechanics. This appears through the electric field arising from charges: $E = q/(4\pi\epsilon_0 r^2)$. This yields: $\epsilon_0 = [-3\ -1\ 4\ 2]$.

2.3 Magnetism

- A magnetic moment has the dimension of a pinpoint magnetic dipole $\boldsymbol{\mu} = I\mathbf{S}$. thus, $[\boldsymbol{\mu}] = [2\ 0\ 0\ 1]$.

- Magnetization is a volume density of magnetic moments: $\mathbf{M} = \boldsymbol{\mu}/V$, so: $[\mathbf{M}] = [-1\ 0\ 0\ 1]$. $\mathbf{M}$ and $\mathbf{H}$ have the same dimension as we can see from: $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$. Thus: $[\mathbf{H}] = [-1\ 0\ 0\ 1]$.

- Magnetic induction B is what matters in Lorentz force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, so that: $[\mathbf{B}] = [0\ 1\ -2\ -1]$.

- Magnetic flux is $\phi = BS$ so that: $[\phi] = [2\ 1\ -2\ -1]$.

- Finally, as in electricity, μ_0 makes the link between the source (current) and fields on one side, and energy and mechanics on the other side, as for the Lorentz force above: $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$, or in vacuum: $\mathbf{curl}\,\mathbf{B} = \mu_0\mathbf{j}$, from which one derives: $[\mu_0] = [1\ 1\ -2\ -2]$.

3 Conversions

3.1 The basis

We will use the expression of a physical quantity in two systems to convert their numerical values in these: $X = X_\alpha \langle X \rangle_\alpha = X_\beta \langle X \rangle_\beta$, where $\langle X \rangle_\alpha$ and $\langle X \rangle_\beta$ are the standard of the dimension of the quantity X in each system. Thus, we need to link one standard to the other to make a link between the numerical values X_α and X_β. To do this, let us start from the elementary dimensions (considered as such in the SI system):

- The standard for length in SI is one meter, which is hundred times larger than the standard in cgs-Gauss, the centimeter. Thus: $\langle \mathrm{L} \rangle_{\mathrm{SI}} = 10^2 \langle \mathrm{L} \rangle_{\mathrm{cgs}}$.
- The standard for mass in SI is one kilogram, which is a thousand times larger than the standard in cgs-Gauss, the gram. Thus: $\langle \mathrm{M} \rangle_{\mathrm{SI}} = 10^3 \langle \mathrm{M} \rangle_{\mathrm{cgs}}$
- Time standards are both equal to one second: $\langle \mathrm{T} \rangle_{\mathrm{SI}} = \langle \mathrm{T} \rangle_{\mathrm{cgs}}$
- The standard of current in SI is 1 A, while it is 10 A in the cgs-Gauss system (the so-called abampere). Thus: $\langle \mathrm{I} \rangle_{\mathrm{SI}} = 10^{-1} \langle \mathrm{I} \rangle_{\mathrm{cgs}}$.

Then, based on the dimensionality derived previously, each standard can be decomposed in the standard of elementary dimensions. Let us take the exemple of force: $F = F_{\mathrm{SI}} \langle F \rangle_{\mathrm{SI}} = F_{\mathrm{cgs}} \langle F \rangle_{\mathrm{cgs}}$. From the dimension $[F] = [1\ 1\ -2\ 0]$ we derive: $\langle F \rangle_{\mathrm{SI}} = \langle \mathrm{L} \rangle_{\mathrm{SI}} \langle \mathrm{M} \rangle_{\mathrm{SI}} \langle \mathrm{T} \rangle_{\mathrm{SI}}^{-2}$, so: $\langle F \rangle_{\mathrm{SI}} = 10^2 \langle \mathrm{L} \rangle_{\mathrm{cgs}} 10^3 \langle \mathrm{M} \rangle_{\mathrm{cgs}} \langle \mathrm{T} \rangle_{\mathrm{cgs}}^{-2}$, and in the end: $\langle F \rangle_{\mathrm{SI}} = 10^5 \langle F \rangle_{\mathrm{cgs}}$, where $\langle F \rangle_{\mathrm{cgs}}$ is called the *dyne*. The latter formula expresses the relationship between the standard units. The relationship between the numerical values is the reverse: $F_{\mathrm{SI}} \langle F \rangle_{\mathrm{SI}} = F_{\mathrm{SI}} \cdot 10^5 \langle F \rangle_{\mathrm{cgs}} = F_{\mathrm{cgs}} \langle F \rangle_{\mathrm{cgs}}$, thus: $F_{\mathrm{cgs}} = F_{\mathrm{SI}} 10^5$. As a consequence, 1 N (for which $F_{\mathrm{SI}} = 1$) is equivalent to 10^5 dyn. The principle is exactly the same for nearly all conversions below, so only the result is provided.

3.2 Mechanics

- Force F. As derived above, 1 N is equivalent to 10^5 dyn.
- Energy $\mathcal{E}$. 1 J is equivalent to 10^7 erg.
- Energy per unit area E_{S}. 1 J/m² is equivalent to 10^3 erg/cm².
- Energy per unit volume E. 1 J/m³ is equivalent to 10 erg/cm³.

3.3 Magnetism

- Induction **B**. 1 T is equivalent to 10^4 G, G standing for *Gauss*.
- Magnetization **M**. 1 A/m is equivalent to 10^{-3} uem/cm³, emu standing for *ElectroMagnetic Unit*.
- Flux ϕ. 1 Wb (Weber) is equivalent to 10^8 Mx, Mx standing for *Maxwell*.
- Moment $\boldsymbol{\mu}$. 1 A · m² is equivalent to 10^3 emu.

We now tackle the case of μ_0. Let us forget for a moment that μ_0 does not show up in cgs-Gauss. From $[\mu_0] = [1\ 1\ -2\ -2]$ one derives: $\mu_{0,\mathrm{SI}} \langle \mu_0 \rangle_{\mathrm{SI}} = \mu_{0,\mathrm{SI}} \cdot 10^7 \langle \mu_0 \rangle_{\mathrm{cgs}} = \mu_{0,\mathrm{cgs}} \langle \mu_0 \rangle_{\mathrm{cgs}}$. Then, as $\mu_{0,\mathrm{SI}} = 4\pi \cdot 10^7$, we conclude that if it existed, $\mu_{0,\mathrm{cgs}} = 4\pi$. This highlights a crucial point on the link between the SI and the cgs-Gauss systems: in the Si system $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$ while in the cgs-Gauss relationship $\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$, it is now clear that the 4π coefficient is simply the numerical value $\mu_{0,\mathrm{cgs}}$ of μ_0, the unit being omitted. It is also clear that the fundamental difference between SI and cgs-Gauss, besides the powers of ten, is a different choice for the definition of the magnetic field: the same physical quantity $\mathbf{B} - \mu_0\mathbf{M}$ is written **H** in cgs-Gauss and $\mu_0\mathbf{H}$ in SI. This is the major source of confusion between the two systems. This said, let us exhibit the conversion factor for the numerical values in the two systems. Following the above statement, we have: $\mu_{0,\mathrm{SI}} \langle \mu_0 \rangle_{\mathrm{SI}} H_{\mathrm{SI}} \langle H \rangle_{\mathrm{SI}} = H_{\mathrm{cgs}} \langle H \rangle_{\mathrm{cgs}}$, from which it follows: $4\pi \cdot 10^{-7} \cdot 10^7 \langle \mu_0 \rangle_{\mathrm{cgs}} H_{\mathrm{SI}} \cdot 10^{-3} \langle H \rangle_{\mathrm{cgs}} = H_{\mathrm{cgs}} \langle H \rangle_{\mathrm{cgs}}$. As $\langle \mu_0 \rangle_{\mathrm{cgs}}$ can be omitted because this quantity is implicit in cgs-Gauss, this relationship becomes: $4\pi \cdot 10^{-3} H_{\mathrm{SI}} = H_{\mathrm{cgs}}$. Thus, for magnetic field **H**, 1 A/m is equivalent to $4\pi \cdot 10^{-3}$ Oe.

As a result of the different definition of H in the two systems, most formulas involving H differ. It is for instance a paradox that dimensionless quantities like susceptibility $\chi = M/H$ and demagnetizing coefficients $H = -NM$ depend on the system of units: $\chi_{\mathrm{SI}} = 4\pi\chi_{\mathrm{cgs}}$ and $\sum N_{\mathrm{cgs}} = 4\pi$. Notice that to avoid this pitfall, some authors define in cgs-Gauss: $\chi = 4\pi M/H$ and $H = -4\pi NM$, and define another demagnetizing coefficient $H = -DM$ with $\sum D_i = 4\pi$. So, when you convert these values, or pick-up values and formulas from the literature, be very careful with the system of units used by the authors and also their notations for χ and N.