Why exchange interaction is a spin-spin interaction and not an interaction between total moments J?

1- L+S = J, and L+2S=g_JJ (g_J : Lande factor), then S= (g_J-1)J
thus $-J_{ij}S_iS_j = -J_{ij}(g_J-1)^2 J_iJ_j$
➔ Exchange interaction between the J_i is: $-(g_J-1)^2 J_{ij}$ (De Gennes factor)

2- The hamiltonian does not contain any term which involves the spin (if no spin-orbit coupling)
For 1 el: wave function expressed in terms of spin and space variables:
$\psi(\vec{r},\sigma) = \chi_\sigma \varphi(\vec{r})$. The orbital part included in $\varphi(\vec{r})$

For 2 electrons the wave functions is again a product: $\Psi(1,2) = \varphi(\vec{r}_1,\vec{r}_2)\chi(\sigma_1,\sigma_2)$

Spin and space appear in the wave function differently: φ includes all space variables. Only φ contains information on L.

Exchange is ∝ $\Delta E_A - \Delta E_S = E(S=1) - E(S=0)$, where $\Delta E_S = \langle \varphi_S | H_{int} | \varphi_S \rangle$

$\Delta E_A = \langle \varphi_A | H_{int} | \varphi_A \rangle$. Orbital degrees of freedom have integrated out