Basic Concepts in Magnetism

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1. Magnetostatics
2. Magnetism of multi-electron atoms
3. Crystal field
4. Magnetism of the free electron gas
5. Dilute magnetic oxides

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www.tcd.ie/Physics/Magnetism
2. Magnetism of multi-electron atoms
2.1 Einstein-de Hass Experiment

Demonstrates the relation between magnetism and angular momentum.

A ferromagnetic rod is suspended on a torsion fibre.

The field in the solenoid is reversed, switching the direction of magnetization of the rod.

An angular impulse is delivered due to the reversal of the angular momentum of the electrons—conservation of angular momentum.

Three huge paradoxes;  — Amperian surface currents
  — Weiss molecular field
  — Bohr - van Leeuwen theorem
The electron

The magnetic properties of solids derive essentially from the magnetism of their electrons. (Nuclei also possess magnetic moments, but they are \( \approx 1000 \) times smaller).

An electron is a point particle with:

- mass: \( m_e = 9.109 \times 10^{-31} \) kg
- charge: \(-e = -1.602 \times 10^{-19} \) C
- intrinsic angular momentum (spin): \( \hbar/2 = 0.527 \times 10^{-34} \) J s

On an atomic scale, magnetism is always associated with angular momentum. Charge is negative, hence the angular momentum and magnetic moment are oppositely directed.
2.2 Origin of Magnetism

At this point it seems that the whole of chemistry and much of physics is understood in principle. The problem is that the equations are much too difficult to solve.....

P. A. M. Dirac
2.3 Orbital and Spin Moment

Magnetism in solids is due to the angular momentum of electrons on atoms.

Two contributions to the electron moment:

• *Orbital motion* about the nucleus

• *Spin* - the intrinsic (rest frame) angular momentum.

\[ m = - \left( \frac{\mu_B}{\hbar} \right)(l + 2s) \]
2.3.1 Orbital moment

Circulating current is $I$; $I = -e/\tau = -ev/2\pi r$

The moment is $m = I A$ $m = -evr/2$

**Bohr:** orbital angular momentum $I$ is quantized in units of $\hbar$; $\hbar$ is Planck’s constant = $6.6226 \times 10^{-34}$ J s; $\hbar = h/2\pi = 1.055 \times 10^{-34}$ J s. $|I| = n\hbar$

**Orbital angular momentum:** $I = m_e r \times v$ Units: J s

**Orbital quantum number** $l$, $l_z = m_l \hbar$ $m_l = 0, \pm 1, \pm 2, \ldots, \pm l$ so $m_z = -m_l(e\hbar/2m_e)$

The Bohr model provides us with the natural unit of magnetic moment

**Bohr magneton** $\mu_B = (e\hbar/2m_e)$ $\mu_B = 9.274 \times 10^{-24}$ A m$^2$ $m_z = m_l \mu_B$

In general $m = \gamma l$ $\gamma = \text{gyromagnetic ratio}$

**Orbital motion** $\gamma = -e/2m_e$

* Derivation can be generalized to noncircular orbits: $m = I A$ for any planar orbit.
The Bohr model also provides us with a natural unit of length, the Bohr radius

\[ a_0 = \frac{4\pi\varepsilon_0 \hbar^2}{m_e e^2} \quad a_0 = 52.92 \text{ pm} \]

And a natural unit of energy, the Rydberg \( R_0 \)

\[ R_0 = \frac{(m/2\hbar^2)(e^2/4\pi\varepsilon_0)^2}{(e^2/4\pi\varepsilon_0)^2} \quad R_0 = 13.606 \text{ eV} \]

\text{\textit{g-factor}}

Ratio of magnitude of \( m \) in units of \( \mu_B \) to magnitude of \( l \) in units of \( \hbar \).

\[ (m/\mu_B) = g(l/\hbar) \]

\( g = 1 \) for orbital motion
2.3.2 Spin moment

Spin is a relativistic effect.

*Spin angular momentum* \( s \)

*Spin quantum number* \( s \)  
\( s = 1/2 \) for electrons

*Spin magnetic quantum number* \( m_s \)
\( m_s = \pm 1/2 \) for electrons

\[ s_z = m_s \hbar \]
\( m_s = \pm 1/2 \) for electrons

\[ m_z = -(e/m_e)m_s \hbar = \pm \mu_B \]

\[ m = -(e/m_e)s \]

For spin moments of electrons we have:

\[ \gamma = -e/m_e \]
\[ g \approx 2 \]

More accurately, after higher order corrections:  
\( g \approx 2.0023 \)  
\( m_z = 1.00116 \mu_B \)

An electron will usually have both orbital and spin angular momentum

\[ m = - (\mu_B/\hbar)(l + 2s) \]
Quantum mechanics of spin

In quantum mechanics, physical observables are represented by operators - differential or matrix.

\( p = -i\hbar \nabla; \quad \text{energy} \; p^2/2m_e = -\hbar^2 \nabla^2 \)

\( n \) magnetic basis states \( \Rightarrow \) \( n \times n \) Hermitian matrix

Spin operator (for \( s = 1/2 \))

\[ s = \sigma \hbar/2 \]

Electron: \( s = 1/2 \) \( \Rightarrow m_s = \pm 1/2 \) i.e spin up and spin down states

Represented by column vectors:

\[ |\uparrow\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |\downarrow\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

Pauli spin matrices

\[ \sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \]

Eigenvalues of \( s^2 \):

\[ s(s+1)\hbar^2 \]

The fundamental property of angular momentum in QM is that the operators satisfy the commutation relations:

\[ [\hat{s}_x, \hat{s}_y] = i\hbar \hat{s}_z, \quad [\hat{s}_y, \hat{s}_z] = i\hbar \hat{s}_x, \quad [\hat{s}_z, \hat{s}_x] = i\hbar \hat{s}_y. \]

\text{or} \quad \hat{s} \times \hat{s} = i\hbar \hat{s}.

Where \([A,B] = AB - BA\) and \([A,B] = 0 \Rightarrow A\) and B’s eigenvalues can be measured simultaneously

\[ [s^2, s_z] = 0 \]
Quantized spin angular momentum of the electron

\[ \frac{1}{2} \leq M_S \leq \frac{1}{2} \]

\[ s = \frac{1}{2} \]

\[ 2\mu_0\mu_B H \]

\[ g\sqrt{s(s+1)}\hbar^2 \]

\[ -\frac{\hbar}{2} \leq M_S \leq \frac{\hbar}{2} \]
The electrons have only two eigenstates, ‘spin up’ ($\uparrow$, $m_s = 1/2$) and ‘spin down’ ($\downarrow$, $m_s = -1/2$), which correspond to two possible orientations of the spin moment relative to the applied field. Populations of the energy levels are given by Boltzmann statistics; $\propto \exp\{-E_i/k_B T\}$. The thermodynamic average $\langle m \rangle$ is evaluated from these Boltzmann populations.

$$
\langle m \rangle = \frac{\mu_B \exp(x) - \mu_B \exp(-x)}{\exp(x) + \exp(-x)} \quad \text{where} \quad x = \mu_0 \mu_B H / k_B T.
$$

In small fields, $\tanh(x) \approx x$, hence the susceptibility $\chi = N\langle m \rangle/H$

$$
\chi = \mu_0 N \mu_B^2 / k_B T
$$

This is again the famous Curie law for the susceptibility, which varies as $T^{-1}$.

In other terms, $\chi = C/T$, where $C = \mu_0 N \mu_B^2 / k_B$ is a constant with dimensions of temperature; Assuming an electron density $N$ of $6 \times 10^{28}$ m$^{-3}$ gives $C \approx 0.5$ K.

The Curie law susceptibility at room temperature is of order $10^{-3}$.
2.4 Spin-Orbit Coupling

Spin and angular momentum coupled to create total angular momentum $j$.

$$m = \gamma j$$

From the electron's point of view, the nucleus revolves round it with speed $v \Rightarrow$ current loop

$$I = Zev/2\pi r$$

Which produces a magnetic field $\mu_0 I/2r$ at the centre

$$B_{so} = \mu_0 Zev/2\pi r^2$$

$$E = - m \cdot B$$

$$E_{so} = - \mu_B B_{so}$$

Since $r \approx a_0/Z$ and $m_e vr \approx \hbar$

$$E_{so} \approx -\mu_0 \mu_B^2 Z^4/4\pi a_0^3$$
2.5 Magnetism of the hydrogenic atom

Orbital angular momentum

The orbital angular momentum operators also satisfy the commutation rules: 
\[ l \times l = \hbar l \]  
and 
\[ [l^2, l_z] = 0 \]

Spherical coordinates

\[ x = r \sin \theta \cos \phi \]
\[ y = r \sin \theta \sin \phi \]
\[ z = r \cos \theta \]

\[ \hat{l}_x = i\hbar (\sin \phi \partial / \partial \theta + \cot \theta \cos \phi \partial / \partial \phi), \]
\[ \hat{l}_y = i\hbar (- \cos \phi \partial / \partial \theta + \cot \theta \sin \phi \partial / \partial \phi), \]
\[ \hat{l}_z = -i\hbar (\partial / \partial \phi). \]

\[ \hat{l}^2 = \hat{l}_x^2 + \hat{l}_y^2 + \hat{l}_z^2 = -\hbar^2 \left( \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) \]
QM operators for orbital angular momentum

**Eigenvalues of** \( l^2 \):

\[ l(l+1)\hbar^2 \]

(l is the orbital angular momentum quantum number)

\( l=1 \) case

\( m_l = 1, 0, -1 \) corresponds to the eigenvectors

\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix},
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]

\( l_x, l_y \) and \( l_z \) operators can be represented by the matrices:

\[
\begin{bmatrix}
0 & 1/\sqrt{2} & 0 \\
1/\sqrt{2} & 0 & 1/\sqrt{2} \\
0 & 1/\sqrt{2} & 0
\end{bmatrix} \hbar,
\begin{bmatrix}
0 & -i/\sqrt{2} & 0 \\
i/\sqrt{2} & 0 & -i/\sqrt{2} \\
0 & i/\sqrt{2} & 0
\end{bmatrix} \hbar,
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix} \hbar
\]

where

\[
\hat{l}^2 = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} 2\hbar^2
\]
Single electron wave functions

**Schroedinger’s equation:**

$$\mathcal{H}\psi_i = \varepsilon_i \psi_i; \quad \mathcal{H} = -\left(\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{1}{\hbar^2 r^2} L^2 \right) - \frac{Ze^2}{4\pi \varepsilon_0 r}$$

Satisfied by the wavefunctions:

$$\psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$$

Where:

$$R(r) = V_n^\ell(Zr/na_0) \exp[-(Zr/na_0)]$$

And the combined angular parts are

$$Y_{s}\ell m_{\ell}(\theta, \phi) \propto P_{s}\ell m_{\ell}(\theta) e^{im_{\ell}\phi}.$$  

(Legendre polynomials)

**Normalized spherical harmonics:**

\[ \begin{align*}
    s & \quad Y_0^0 = \sqrt{1/4\pi} \\
    p & \quad Y_1^0 = \sqrt{3/4\pi} \cos \theta \\
    d & \quad Y_2^0 = \sqrt{5/16\pi} (3\cos^2 \theta - 1) \\
    f & \quad Y_3^0 = \sqrt{7/16\pi} (5\cos^3 \theta - 3\cos \theta) \\
    p & \quad Y_1^\pm = \pm \sqrt{3/8\pi} \sin \theta \exp(\pm i\phi) \\
    d & \quad Y_2^\pm = \pm \sqrt{15/8\pi} \sin \theta \cos \theta \exp(\pm i\phi) \\
    f & \quad Y_3^\pm = \pm \sqrt{21/64\pi} (5\cos^2 \theta - 1) \sin \theta \exp(\pm i\phi) \\
\end{align*} \]
The three quantum number \( n, l, m_l \) denote an orbital.
Orbitals are denoted \( n_x m_l \), \( x = s, p, d, f \ldots \) for \( l = 0, 1, 2, 3, \ldots \)
Each orbital can accommodate at most two electrons* \((m_s = \pm 1/2)\)

**The hydrogenic orbitals:** An orbital can accommodate \( 2(2l+1) \) electrons.

<table>
<thead>
<tr>
<th></th>
<th>( n )</th>
<th>( l )</th>
<th>( m_l )</th>
<th>( m_s )</th>
<th>No of states</th>
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<tr>
<td>1s</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( \pm 1/2 )</td>
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</tr>
<tr>
<td>2s</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>( \pm 1/2 )</td>
<td>2</td>
</tr>
<tr>
<td>2p</td>
<td>2</td>
<td>1</td>
<td>0, ( \pm 1 )</td>
<td>( \pm 1/2 )</td>
<td>6</td>
</tr>
<tr>
<td>3s</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>( \pm 1/2 )</td>
<td>2</td>
</tr>
<tr>
<td>3p</td>
<td>3</td>
<td>1</td>
<td>0, ( \pm 1 )</td>
<td>( \pm 1/2 )</td>
<td>6</td>
</tr>
<tr>
<td>3d</td>
<td>3</td>
<td>2</td>
<td>0, ( \pm 1, \pm 2 )</td>
<td>( \pm 1/2 )</td>
<td>10</td>
</tr>
<tr>
<td>4s</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td>( \pm 1/2 )</td>
<td>2</td>
</tr>
<tr>
<td>4p</td>
<td>4</td>
<td>1</td>
<td>0, ( \pm 1 )</td>
<td>( \pm 1/2 )</td>
<td>6</td>
</tr>
<tr>
<td>4d</td>
<td>4</td>
<td>2</td>
<td>0, ( \pm 1, \pm 2 )</td>
<td>( \pm 1/2 )</td>
<td>10</td>
</tr>
<tr>
<td>4f</td>
<td>4</td>
<td>3</td>
<td>0, ( \pm 1, \pm 2, \pm 3 )</td>
<td>( \pm 1/2 )</td>
<td>14</td>
</tr>
</tbody>
</table>

*The Pauli exclusion principle:* No two electrons can have the same four quantum number.
Hydrogenic orbitals

\[ m_\ell = 0 \]
\[ \ell = 0 \]

\[ m_\ell = 0 \]
\[ \ell = 1 \]

\[ m_\ell = 0 \]
\[ \ell = 2 \]

\[ m_\ell = 1 \]
\[ \ell = 1 \]

\[ m_\ell = 1 \]
\[ \ell = 2 \]

\[ m_\ell = 2 \]
2.7 The Many Electron Atom

\[ \mathcal{H}_0 = \sum_i \left[ -\frac{\hbar^2}{2m_e} \nabla_i^2 - \frac{Ze^2}{4\pi \epsilon_0 r_i} \right] + \sum_{i<j} \frac{e^2}{4\pi \epsilon_0 r_{ij}}. \]

**Hartree-Foch approximation**

- No longer a simple Coulomb potential.
- \(l\) degeneracy is lifted.
- Solution: Suppose that each electron experiences the potential of a different spherically-symmetric potential.
Addition of angular momentum

First add the orbital and spin momenta \( l_i \) and \( s_i \) to form \( L \) and \( S \). Then couple them to give the total \( J \)

\[ J = L + S \]

\(|L-S| \leq J \leq |L+S|\)

Different \( J \)-states are termed *multiplets*.

Denoted by:

\[ ^{2S+1}X_J \]

\( X = S, P, D, F, ... \) for \( L = 0,1,2,3,... \)

**Hund’s rules**

For determining the ground-state of a multi-electron atom/ion.

1) Maximize \( S \)
2) Maximize \( L \) consistent with \( S \).
3) Couple \( L \) and \( S \) to form \( J \).
   - Less than half full shell \( J = L-S \)
   - Exactly half full shell \( J = S \)
   - More than half full shell \( J = L+S \)
Hund’s rules
Examples of Hund’s rules

<table>
<thead>
<tr>
<th>Ion</th>
<th>Configuration</th>
<th>Spin Up</th>
<th>Spin Down</th>
<th>Total</th>
<th>Term</th>
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<tbody>
<tr>
<td>Fe$^{3+}$</td>
<td>3d$^5$</td>
<td>↑↑↑↑↑↑↑↑</td>
<td>ooooo</td>
<td></td>
<td>$^6S_{5/2}$</td>
</tr>
<tr>
<td></td>
<td>S = 5/2  L = 0</td>
<td>J = 5/2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ni$^{2+}$</td>
<td>3d$^8$</td>
<td>↑↑↑↑↑↑↓↓↓↓</td>
<td>oo</td>
<td></td>
<td>$^3F_4$</td>
</tr>
<tr>
<td></td>
<td>S = 1  L = 3</td>
<td>J = 4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nd$^{3+}$</td>
<td>4f$^3$</td>
<td>↑↑↑↑↑↑↑↑↑↑</td>
<td>looooooo</td>
<td></td>
<td>$^4I_{9/2}$</td>
</tr>
<tr>
<td></td>
<td>S = 3/2 L = 6</td>
<td>J = 9/2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dy$^{3+}$</td>
<td>4f$^9$</td>
<td>↑↑↑↑↑↑↑↑↓↓↓↓</td>
<td>oooooo</td>
<td></td>
<td>$^6H_{15/2}$</td>
</tr>
<tr>
<td></td>
<td>S = 5/2 L = 5</td>
<td>J = 15/2</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>
2.8 Spin-Orbit Coupling

\[ \mathcal{H}_{so} = \Lambda L \cdot S \]

\( \Lambda \) is the spin-orbit coupling constant

\( \Lambda > 0 \) for the 1st half of the 3\( d \) or 4\( f \) series.
\( \Lambda < 0 \) for the 2nd half of the 3\( d \) or 4\( f \) series.

Single-electron atom case:

\[ \mathcal{H}_{so} = \lambda l \cdot s \]

\[ \Lambda = \pm (1/2) \lambda / 2 S \]

\[ L \cdot S = (1/2)(J^2 - L^2 - S^2) = (\hbar^2/2)[J(J+1)-L(L+1)-S(S+1)] \]
2.9 Zeeman Interaction

\[ \mathbf{m} = - \left( \frac{\mu_B}{\hbar} \right) (L+2S) \]

The energy of a moment in a magnetic field is:

\[ E = - \mathbf{m} \cdot \mathbf{B} \]

Hence:

\[ \mathcal{H}_Z = \left( \frac{\mu_B}{\hbar} \right) (L+2S) \cdot \mathbf{B} \]

Lande g-factor

\[ g = - \left( \frac{\mathbf{m} \cdot \mathbf{J}}{\mu_B} / \frac{\mathbf{J}^2}{\hbar} \right) \]

Eigenvalues of \( \mathbf{J}^2 \) are: \( J(J+1) \hbar^2 \)

\[ \mathbf{m} \cdot \mathbf{J} = -\left( \frac{\mu_B}{\hbar} \right) [(L + 2S) \cdot (L + S)] \]

\[ = -\left( \frac{\mu_B}{\hbar} \right) [(L^2 + 3L \cdot S + 2S^2)] \text{ since } L \text{ and } S \text{ commute} \]

\[ = -\left( \frac{\mu_B}{\hbar} \right) [(L^2 + 2S^2 + (3/2)(J^2 - L^2 - S^2)] \]

\[ = -\left( \frac{\mu_B}{\hbar} \right) [(3/2)J(J+1) - (1/2)L(L+1) + (1/2)S(S+1)] \]

\[ g = (3/2) + [S(S+1) - L(L+1)] / 2J(J+1) \]

\[ \mathbf{m} = - g \mu_B \mathbf{J} / \hbar \]
Example: Co$^{2+}$ free ion

The energy levels for a free ion with electronic configuration $3d^7$: Co$^{2+}$

$L = 3, S = \frac{3}{2}, J = \frac{9}{2}, g = \frac{5}{2}$. 

$L = 1, S = \frac{3}{2}, J = L - S = \frac{3}{2}$

$L = 3, S = \frac{3}{2}, J = L + S = \frac{9}{2}$. 

$M_J = \{\frac{3}{2}, \frac{-3}{2}, \frac{5}{2}, \frac{-5}{2}, \frac{7}{2}, \frac{-7}{2}, \frac{9}{2}, \frac{-9}{2}\}$. 
Energy levels of an ion with $J = 5/2$ in an applied field
2.10 Curie Law Susceptibility

**Curie law** \[ \chi = \frac{C}{T} \]

- \( C \) is Curie’s constant.
- Units: Kelvin, K.
- Typical values \( \sim 1 \text{K} \)

The thermodynamic average of the moment:

\[
\langle m \rangle = \frac{\sum_i m_i \exp(-\varepsilon_i/k_BT)}{\sum_i \exp(-\varepsilon_i/k_BT)}.
\]

Using the identities:

\[
B = Bz \quad E = -m \cdot B \quad \Rightarrow \quad \langle m_z \rangle = \frac{\sum_{-J}^{J} -g \mu_B M_J (1 - \mu_0 g \mu_B M_J H/k_BT)}{\sum_{-J}^{J} (1 - \mu_0 g \mu_B M_J H/k_BT)}.
\]

Using the identities:

\[
\sum_{-J}^{J} 1 = 2J + 1; \quad \sum_{-J}^{J} M_J = 0 \quad \sum_{-J}^{J} M_J^2 = J(J+1)(2J+1)/3,
\]

And the fact that \( \chi = n \langle m \rangle / H \) (\( n \) is the number density of atoms/ions)

We get that

\[
C = \frac{\mu_0 ng^2 \mu_B^2 J(J+1)}{3k_B}.
\]
4f ions

<table>
<thead>
<tr>
<th>4f&lt;sup&gt;n&lt;/sup&gt;</th>
<th>S</th>
<th>L</th>
<th>J</th>
<th>g</th>
<th>m&lt;sub&gt;0&lt;/sub&gt; = gJ</th>
<th>m&lt;sub&gt;eff&lt;/sub&gt; = g√J(J + 1)</th>
<th>m&lt;sub&gt;eff&lt;/sub&gt;&lt;sup&gt;exp&lt;/sup&gt;</th>
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<td>1/2</td>
<td>3</td>
<td>5/2</td>
<td>6/7</td>
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<td>2.5</td>
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<td>1</td>
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<td>3/2</td>
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3d ions

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<th>$3d^n$</th>
<th>$S$</th>
<th>$L$</th>
<th>$J$</th>
<th>$g$</th>
<th>$m_{\text{eff}} = g \sqrt{J(J+1)}$</th>
<th>$m_{\text{eff}} = g \sqrt{S(S+1)}$</th>
<th>$m_{\text{exp}}^{\text{eff}}$</th>
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<td>Ti$^{3+}$, V$^{4+}$</td>
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<td>2</td>
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<td>2.83</td>
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<td>8</td>
<td>Ni$^{3+}$</td>
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<td>$\frac{6}{5}$</td>
<td>3.55</td>
<td>1.73</td>
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</table>
b) Magnetization

To calculate the complete magnetization curve, set \( y = g\mu_B\mu_0 H/k_B T \), then

\[
<m> = g\mu_B \frac{\partial}{\partial y} \ln \sum_j e^{Jy} \exp\{M_j y\} \\
[d(ln z)/dy = (1/z) dz/dy]
\]

The sum over the energy levels must be evaluated; it can be written as

\[
\exp(Jy) \{1 + r + r^2 + \ldots + r^{2J}\} \text{ where } r = \exp\{-y\}
\]

The sum of a geometric progression \( (1 + r + r^2 + \ldots + r^n) = (r^{n+1} - 1)/(r - 1)\)

\[
\therefore \sum_j e^{M_j y} = (\exp\{-(2J+1)y\} - 1)\exp\{Jy\}/(\exp\{-y\}-1)
\]

multiply top and bottom by \( \exp\{y/2\} \)

\[
= [\sinh(2J+1)y/2]/[\sinh y/2]
\]

\[
<m> = g\mu_B (\partial/\partial y)\ln\{[\sinh(2J+1)y/2]/[\sinh y/2]\}
\]

\[
= g\mu_B/2 \{(2J+1)\coth(2J+1)y/2 - \coth y/2\}
\]

setting \( x = Jy \), we obtain

\[
<m> = m B_J(x)
\]

where the Brillouin function

\[
B_J(x) = \{(2J+1)/2\}\coth(2J+1)x/2 - (1/2J)\coth(x/2J).
\]

This reduces to \( <m> = \mu_B \tanh(x) \)

in the limit \( J = 1/2, g = 2 \).
Comparison of the Brillouin functions for $s = 1/2$, $S = 2$ and the Langevin function ($S = \infty$)
Reduced magnetization curves of three paramagnetic salts, compared with Brillouin function predictions.
Basic Concepts in Magnetism

J. M. D. Coey

School of Physics and CRANN, Trinity College Dublin

Ireland.

1. Magnetostatics
2. Magnetism of multi-electron atoms
3. Crystal field
4. Magnetism of the free electron gas
5. Dilute magnetic oxides

Comments and corrections please: jcoey@tcd.ie  www.tcd.ie/Physics/Magnetism